# Czech-Polish-Slovak Match

## IST Austria, 24 – 27 June 2018

**1.** *Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that, for all real numbers $x$ and $y$,*

$$f(x^2 + xy) = f(x)f(y) + yf(x) + xf(x + y).$$

(*Walther Janous, Austria*)

**Solution.** Setting $x = 0$ yields $f(0) = f(0)f(y) + yf(0)$. If $f(0) \neq 0$, we obtain $1 = f(y) + y$ or equivalently $f(y) = 1 - y$ for all $y \in \mathbb{R}$. Inserting this in the original equation yields

$$1 - x^2 - xy = (1 - x)(1 - y) + y(1 - x) + x(1 - x - y)$$

for $x$, $y \in \mathbb{R}$, which is true.

Therefore, we are left with $f(0) = 0$.

With the substitution $x + y = z$, the functional equation is equivalent to

$$f(xz) = f(x)f(z - x) + (z - x)f(x) + xf(z). \tag{1}$$

Exchanging $x$ and $z$ yields

$$f(zx) = f(z)f(x - z) + (x - z)f(z) + zf(x). \tag{2}$$

Combining (1) and (2) yields

$$f(x)f(z - x) + (z - x)f(x) + xf(z) = f(z)f(x - z) + (x - z)f(z) + zf(x)$$

or equivalently

$$f(x)\Big(f(z - x) - x\Big) = f(z)\Big(f(x - z) - z\Big). \tag{3}$$

Setting $z = 0$ yields

$$f(x)(f(-x) - x) = f(0)f(x) = 0,$$

so for each $x \in \mathbb{R}$ we either have $f(x) = 0$ or $f(-x) = x$.

Assume that there is an $x \neq 0$ with $f(x) = 0$. For $z \neq 0$, we have $f(x - z) \in \{0, z - x\}$, so $f(x - z) \neq z$. Thus (3) implies $f(z) = 0$ for all $z \neq 0$. It is clear that the constant function $f = 0$ is a solution.

Otherwise, we have $f(x) \neq 0$ for all $x \neq 0$ and therefore $f(x) = -x$ for all $x$. This is also a solution.

We conclude that there are three solutions $f$, namely $f = 0$, $f(x) = -x$ and $f(x) = 1 - x$.

**Alternative Solution.** Setting $x = 0$ gives $f(0) = f(0)f(y) + yf(0)$. Since $f(y) = 1 - y$ gives a solution, we remain with the case $f(0) = 0$. Setting $x = 1$ gives

$$f(1 + y) = f(1)f(y) + yf(1) + f(1 + y), \qquad \text{i. e.} \qquad 0 = f(1)(f(y) + y).$$

Since $f(y) = -y$ is a solution, we remain with the case $f(1) = 0$.

Now suppose $f(0) = f(1) = 0$. Setting $y = -x$ gives $0 = f(x)f(-x) - xf(x)$, that is

$$f(x) = 0 \qquad \vee \qquad f(-x) = x. \tag{4}$$

Setting $y = 1 - x$ gives $f(x) = f(x)f(1-x) + (1-x)f(x)$, that is

$$f(x) = 0 \qquad \vee \qquad f(1-x) = x. \tag{5}$$

If $f(x) \neq 0$ for some $x \neq 0$, we have $f(-x) = x \neq 0$ and $f(1-x) = x \neq 0$ and applying (4) and (5) to $-x$ and $1-x$ instead of $x$ we get $f(x) = -x = 1-x$, a contradiction. Hence $f(x) = 0$ for all $x \in \mathbb{R}$.

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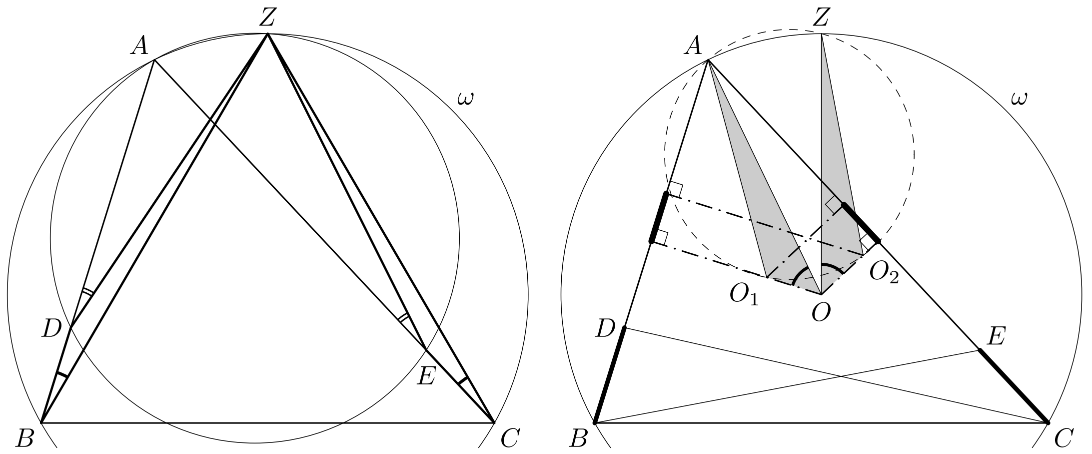
**2.** *Let $ABC$ be an acute scalene triangle. Let $D$ and $E$ be points on the sides $AB$ and $AC$, respectively, such that $BD = CE$. Denote by $O_1$ and $O_2$ the circumcentres of the triangles $ABE$ and $ACD$, respectively. Prove that the circumcircles of the triangles $ABC$, $ADE$ and $AO_1O_2$ have a common point different from $A$.*

(*Patrik Bak, Slovakia*)

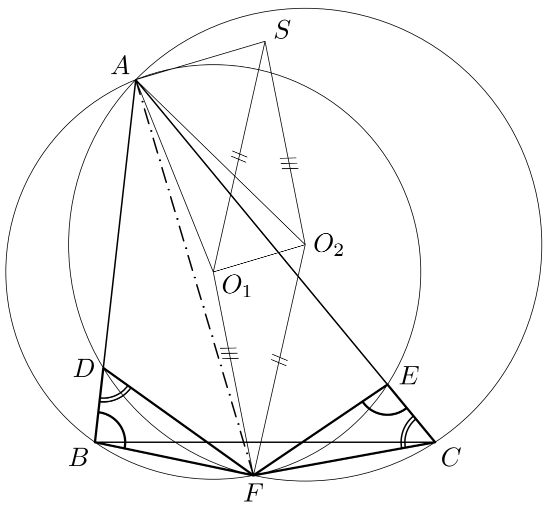
**Solution 1.** Let $Z$ be the midpoint of the longer arc $BC$ of the circumcirle $\omega$ of the triangle $ABC$. The triangles $ZDB$ and $ZEC$ are congruent, because they agree in the sides $BD = CE$ and $ZB = ZC$, as well as in the corresponding angles between them, for both lie over the chord $AZ$ of $\omega$. It follows that $\angle ZDA = \angle ZEA$, which in turn discloses that the quadrilateral $ADEZ$ is cyclic. So it remains to be shown that the quadrilateral $AO_1O_2Z$ is cyclic.

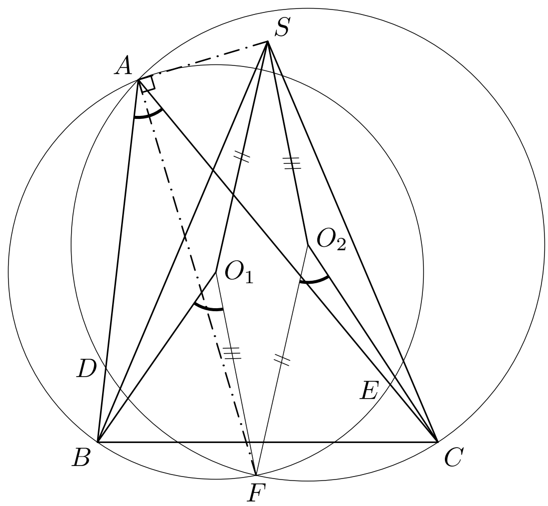



The center $O$ of $\omega$ satisfies $OO_1 \perp AB$ and $OO_2 \perp AC$. The projections of $O$ and $O_1$ onto $AC$ are the midpoints of $AC$ and $AE$ respectively. Thus the projection of the segment $OO_1$ onto $AC$ has length $\frac{1}{2}CE$. For the same reason, the projection of $OO_2$ on $AB$ has length $\frac{1}{2}BD$, and by hypothesis these two length agree. Moreover, the angle between $OO_1$ and $AC$ is the same as the angle between $OO_2$ and $AB$. It follows that $OO_1 = OO_2$.

Further, we have $\angle AOO_1 = \angle ACB = \angle O_2OZ$, the latter being a consequence of $ZO \perp BC$ and $OO_2 \perp AC$. So the rays $OA$ and $OZ$ are isogonal in the angle $O_2OO_1$. In the combination with $AO = ZO$ and $OO_1 = OO_2$ this proves that the quadrilateral $AO_1O_2Z$ is an isocleses trapezium and thus in particular cyclic. Thereby the problem is solved.

**Solution 2.** Let the circumcircles of triangles $ABE$ and $ADC$ intersect each other again at $F \not\equiv A$. Then the triangles $BFD$ and $EFC$ are congruent, for they agree in their sides $BD = CE$ as well as in their corresponding adjacent angles, i. e., $\angle FBD = \angle FEC$ and $\angle BDF = \angle ECF$. It follows that the altitudes of these triangles passing through $F$ have the same lengths, wherefore $AF$ is the bisector of the angle $BAC$.




Now construct the point $S$ such that $SO_1FO_2$ is a parallelogram. We will show that $S$ is the desired point.

To prove that $S$ lies on the circumcircle of triangle $AO_1O_2$, we note that the triangles $AO_1O_2$ and $FO_1O_2$ are congruent due to $AO_1 = FO_1$ and $AO_2 = FO_2$. It follows that $AO_1O_2S$ is an isosceles trapezium and hence in particular a cyclic quadrilateral, as claimed. Later, it will help us to have observed that the facts used in this paragraph imply $AS \parallel O_1O_2 \perp AF$.

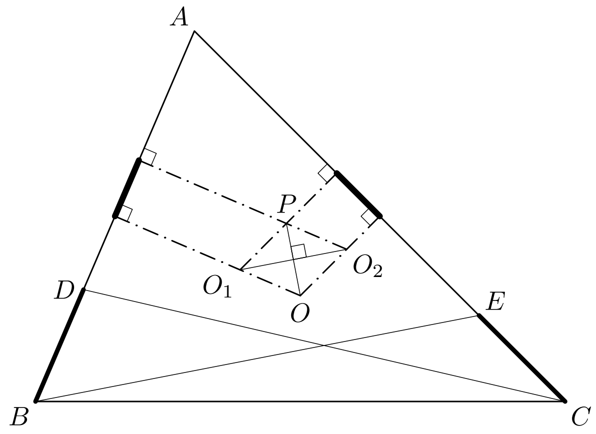
Next, we prove that $S$ lies on the circumcircle of the triangle $ABC$ and that it is actually the midpoint of its longer arc $BC$; this will also show $S \not\equiv A$, as needed. Our first intermediate step is to observe that the triangles $O_1SB$ and $O_2CS$ are congruent. Indeed they agree in a pair of sides, $BO_1 = FO_1 = SO_2$ and $SO_1 = FO_2 = CO_2$. Moreover the corresponding angles between these sides are equal, because their complements to 360° are equal as a consequence of

$$\angle BO_1F = 2\angle BAF = 2\angle FAC = \angle FO_2C$$

and $\angle FO_1S = \angle SO_2F$. This concludes the verification of $\triangle O_1SB \cong \triangle O_2CS$, and it follows that $BS = CS$. Further, since $AF$ is the bisector of $\angle BAC$ and $AF \perp AS$, the line $AS$ is the exterior bisector of $\angle BAC$. Altogether we obtain that $S$ is the point described above. The fact that $S$ lies on the circumcircle of the triangle $ADE$ can be shown as in the first solution.

**Solution 3.** Denote by $O$ and $P$ the circumcentres of triangles $ABC$ and $ADE$, respectively. The lines $OO_1$ and $O_2P$ (being the perpendicular bisectors of $AB$

and $AD$, respectively) are both perpendicular to $AB$ and their distance is $\frac{1}{2}BD$. Similarly, the lines $OO_2$ and $O_1P$ are both perpendicular to $AC$ and their distance is $\frac{1}{2}CE$. Since $\frac{1}{2}BD = \frac{1}{2}CE$, the quadrilateral $O_1OO_2P$ is a parallelogram with equal altitudes, hence a rhombus. It follows that $OP$ is the perpendicular bisector of $O_1O_2$, so all the three circumcentres of the triangles $ABC$, $ADE$ and $AO_1O_2$ lie on the same line, which concludes the claim (since $A$ does not lie on this line because of $AB \neq AC$).

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**3.** *There are 2018 players sitting around a round table. At the beginning of the game we arbitrarily deal all the cards from a deck of $K$ cards to the players (some players may receive no cards). In each turn we choose a player who draws one card from each of the two neighbours. It is only allowed to choose a player whose each neighbour holds a nonzero number of cards. The game terminates when there is no such player. Determine the largest possible value of $K$ such that, no matter how we deal the cards and how we choose the players, the game always terminates after a finite number of turns.* (Peter Novotný, Slovakia)

**Solution.** The answer is $K = 2017$.

For $K = 2018$, we deal 2 cards to one player, 0 cards to one of his neighbours and 1 card to everyone else. Then in each turn we choose the player with 0 cards:

$$\dots 1\,1\,\underline{2\,0\,1}\,1\,1 \dots \qquad \to \qquad \dots 1\,1\,\underline{1\,2\,0}\,1\,1 \dots$$

After each turn, the configuration stays the same – there is one player with 2 cards, one of his neighbours with 0 cards and all the others with 1 card (the only change is that the positions of the players with 2 and 0 cards is shifted). Therefore we can make moves forever and the game never terminates.

Whenever $K > 2018$, we can play forever using the same strategy as for $K = 2018$. We simply deal the extra cards arbitrarily and ignore them during the game.

Now we will prove that for $K = 2017$ the game terminates no matter how we play. Let us call *zeros* the players with no cards and *ones* the players with exactly one card. The zeros split the other players into *segments* of various lengths. When two zeros sit next to each other, they form a segment with a length of 0. Also note that there is obviously at least one zero when $K = 2017$.

*Lemma.* There exists a segment containing no other players than ones (possibly with a length of 0).

*Proof.* If we add to each segment the zero which bounds it in the clock-wise direction, then the sum of the lengths of all the segments will be 2018. There are only 2017 cards, therefore at least one segment contains fewer cards than players, which is possible only when all the players of this segment, except for the bounding zero, are ones. $\square$

Let us consider the shortest segment among the ones containing no other players than ones; the lemma assures the existence of such a segment. If we choose a zero adjacent to this segment, we shorten it by 1 (or by 2 — in the special case when there is exactly one zero in the game):

$$\dots\underline{*\,0\,1}\,1\,1\dots1\,0\dots \qquad\to\qquad \dots\underline{*\,2\,0}\,1\,1\dots1\,0\dots$$

If we choose one of the ones inside of the shortest segment, we create two even shorter segments:

$$\dots0\,1\dots1\,\underline{1\,1\,1}\,1\dots1\,0\dots \qquad\to\qquad \dots0\,1\dots1\,\underline{0\,3\,0}\,1\dots1\,0\dots \qquad\to$$

The length of the shortest segment could decrease only finitely many times. From the moment when it stops decreasing we won't be able to choose any of the zeros bounding the shortest segment, nor any of the ones inside of it. This means that the game will continue on the other side of the table between the bounding zeros of the shortest segment. The neighbours of these two zeros won't be able to get any more cards, so we cannot choose them anymore. The neighbours of these neighbours will thereby be chosen at most finitely many times (at most the number of times equal to the number of cards of these neighbours), so after some time we won't be able to choose them. We can use this reasoning repeatedly. The part of the table where we still can choose players eventually decreases, which means that the game cannot last infinitely long.

**Remark.** If $K = 2018$ and we give one card to every player, then after one move we would get a segment of ones bounded by two zeros. In that case the game necessarily ends after finitely many moves (to see it we just need to use the reasoning from the solution).

**Remark.** As soon as we show that the game will be played only in one part of the table bounded by two players (so no cards will ever pass some line of the table and therefore it could be think of as a line segment), we might just use a right mono-variant to prove that the game is finite. For example, to each card we might assign its distance to one of the bounds and keep track of the sum of squares of these distances. In each move this number is decreased by

$$(a-1)^2 + (a+1)^2 - 2a^2 = 2,$$

and since it cannot be negative, the game will have to eventually end.

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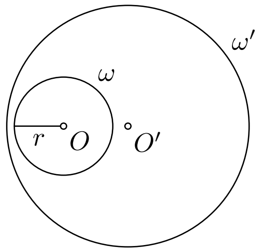
**4.** *Let $ABC$ be an acute triangle with the perimeter of $2s$. We are given three pairwise disjoint circles with pairwise disjoint interiors with the centres $A$, $B$ and*

*$C$, respectively. Prove that there exists a circle with the radius of $s$ which contains all the three circles.* (*Josef Tkadlec, Czechia*)

**Solution 1.** To simplify the formulations, we say that a point lies inside of the circle if it lies on that circle or in its interior. Assume we are given a circle $\omega$ with the radius of $r$ and the centre $O$. A circle $\omega'$ with the centre $O'$ contains the circle $\omega$ if and only if its radius is at least $O'O + r$.

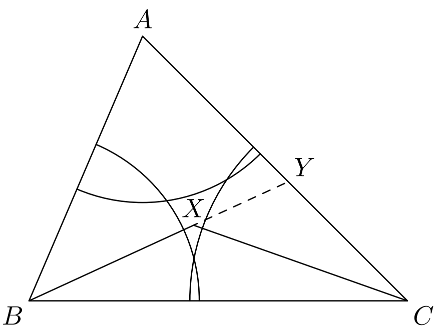



Denote by $r_a$, $r_b$, $r_c$ the radii of our circles with the centres at $A$, $B$ and $C$, respectively. Using our observation three times indicates that the centre $X$ of the circle we are seeking has to meet $s \geq AX + r_a$, or equivalently $AX \leq s - r_a$, and analogously $BX \leq s - r_b$ and $CX \leq s - r_c$.

Notice that the numbers $s - r_a$, $s - r_b$ and $s - r_c$ are positive. We will show this for $s - r_a$. Since our circles are disjoint with disjoint interiors, we know that $r_a < b$ and $r_a < c$. This gives us $r_a < (b + c)/2 < (a + b + c)/2 = s$, which indeed means that $s - r_a$ is a positive number.

Now we may consider three circles with the centres $A, B$ and $C$ and radii $s - r_a$, $s - r_b$ and $s - r_c$, respectively. If we prove that there is a point $X$ lying inside each of them, we will be done.

Each two of these three circles intersect at two points, because for example $(s - r_a) + (s - r_b) > 2s - c = a + b > c$ (and also $c > |(s - r_a) - (s - r_b)|$). For the sake of contradiction assume there is no point lying inside all of them. Then the situation looks like on the picture, that is, there exists a point $X$ inside of the triangle which lies outside of the three circles (see the remark at the end):

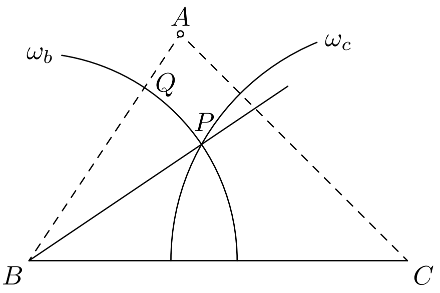



For such $X$ we have $AX + BX + CX > s - r_a + s - r_b + s - r_c > 2s$. This is not possible, however. Let $Y$ be the intersection of $BX$ and $AC$. Then usign the triangle inequalities for the triangles $CXY$, $ABY$ we get

$$BX + CX < BX + XY + CY = BY + CY < AB + AY + CY = AB + AC.$$

Similarly $AX + BX < AC + BC$ and $CX + AX < BC + AB$. Summing these three inequalities we obtain $AX + BX + CY < AB + BC + AC = 2s$, which is a contradiction.

**Remark.** If three circles $\omega_a$, $\omega_b$ and $\omega_c$ with the centres $A$, $B$ and $C$, respectively, satisfy the conditions that each two of them intersect and there is no point lying inside all of the three circles, then there exists a point in the interior of the triangle $ABC$ which lies outside of each of the three circles.




To prove this, consider the intersection point $P$ of $\omega_b$ and $\omega_c$ which lies in the halfplane determined by the line $BC$ and the point $A$. The intersection $Q$ of the ray $BA$ with $\omega_b$ lies inside of $\omega_a$, since it is the closest point of $\omega_b$ to $A$ (this is true even if $A$ is inside of $\omega_b$, since $\omega_a \cap \omega_b \neq \emptyset$). Therefore $A$ cannot lie in the angle $CBP$ (otherwise $Q$ would lie inside all of the three circles). But that means $P$ lies in the interior of the angle $CBA$. Similarly $P$ lies in the interior of the angle $BCA$. So we have that $P$ lies in the interior of the triangle $ABC$. Since $P$ does not lie inside of $\omega_a$, there is a point in the neigbourhood of $P$ lying outside of all the three circles.

**Solution 2.** (by *Tomáš Sásik.*) We will use the same notation for the radii of the given circles. Also here, we will consider the circles $\psi_a$, $\psi_b$, $\psi_c$ with radii $s - r_a$, $s - r_b$, $s - r_c$ and prove that they have a common point. Without loss of generality assume that $AB$ is the shortest side. Let $a = BC$, $b = CA$, $c = AB$. Because of the disjunction we have $r_a + r_b < c$, so $s - r_a + s - r_b > 2s - c = a + b$. Therefore at least one of the inequalities

$$s - r_a > b, \qquad s - r_b > a$$

must be true and the point $C$ lies inside of at least one of the circles $\psi_a$, $\psi_b$. Without loss of generality we may assume it lies in $\psi_a$. Since $AB \leq AC$, we also have that $B$ lies inside of $\psi_a$, therefore the whole triangle $ABC$ lies there.

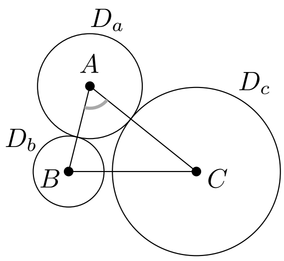
We have $s - r_b + s - r_c \geq 2s - a > 2s - b - c = a$, so the circles $\psi_b$, $\psi_c$ have a common point on the side $BC$, which lies also inside of the circle $\psi_a$. The rest follows as in the first solution.

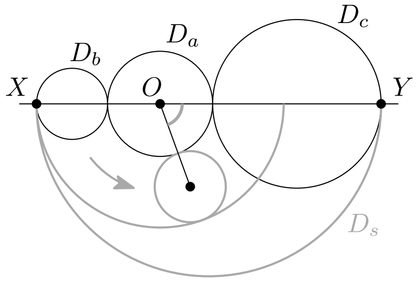
**Solution 3.** (by *Radek Olšák.*) First we grow the disks until two pairs of them become tangent. Denote the new discs centered at $A$, $B$, $C$ by $D_a$, $D_b$, $D_c$ and their radii by $r_a$, $r_b$, $r_c$. Without loss of generality, assume that $D_a$ is tangent to both $D_b$ and $D_c$ and that $r_b \leq r_c$. Clearly, $r_a + r_b + r_c \leq s$.

Consider a different configuration of the three discs $D_b$, $D_a$, $D_c$ in which their centers lie on a line in this order and the neighbouring disks are tangent. Let $X$, $Y$

be as in the figure and let $D_s$ be a disc with diameter $XY$. Clearly, $D_s$ covers all three smaller discs and since $XY = 2(r_a + r_b + r_c) \leq 2s$, its radius is at most $s$.

Finally, we observe that since $r_b \leq r_c$, we can rotate $D_b$ around the center $O$ of $D_a$ and it stays sinside $D_s$ (the disk with center $O$ and radius $r_a + 2r_b$ is contained inside $D_s$). Hence the original configuration of disks can be covered by a disk of radius at most $s$ as well.

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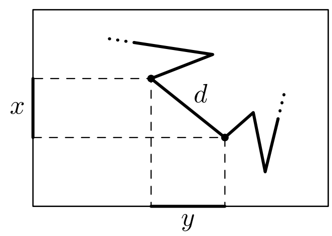
**5.** *In a rectangle with dimensions $2 \times 3$ there is a polyline of length 36, which can have self-intersections. Show that there exists a line parallel to two sides of the rectangle, which intersects the other two sides in their interior points and intersects the polyline in fewer than 10 points.*

(*Josef Tkadlec, Czechia, Vojtech Bálint, Slovakia*)




**Solution.** Consider an arbitrary line segment of the polyline and denote by $d$ its length and by $x$ and $y$ the lengths of its perpendicular projections on the sides of lengths 2 and 3, respectively. Cauchy-Schwarz inequality gives us

$$(2x + 3y)^2 \leq (2^2 + 3^2)(x^2 + y^2) = 13d^2,$$

which means $2x + 3y \leq d \cdot \sqrt{13}$. Denote by $X$ and $Y$ the total length of all the perpendicular projections of all the line segments on the sides of lengths 2 and 3, respectively. Summing up our estimations for each line segment gives us $2X + 3Y \leq 36 \cdot \sqrt{13} < 130$. But then either $2X < 40$, or $3Y < 90$. In the first case we would have $X < 20$, so on the side of length 2 there is a point that is contained in fewer than 10 projections. A line perpendicular to this side at this point intersects the polyline at most 9 times. The other case is analogous.

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**6.** *We say that a positive integer $n$ is fantastic, if there exist positive rational numbers $a$ and $b$ such that*

$$n = a + \frac{1}{a} + b + \frac{1}{b}.$$

(a) *Prove that there exist infinitely many prime numbers $p$ such that no multiple of $p$ is fantastic.*

(b) *Prove that there exist infinitely many prime numbers $p$ such that some multiple of $p$ is fantastic.*

(*Walther Janous, Austria*)

**Solution.** Note that

$$r(a,b) := a + \frac{1}{a} + b + \frac{1}{b} = \frac{(a+b)(ab+1)}{ab}.$$

We put $a = \frac{t}{u}$ and $b = \frac{v}{w}$, where $t$, $u$, $v$ and $w$ are positive integers such that both $t$ and $u$ and also $v$ and $w$ are coprime. Then we get $r(a,b) = \frac{(tv+uw)(tw+uv)}{tuvw}$, whence the Diophantine equation

$$tu(v^2 + w^2) + vw(t^2 + u^2) = kptuvw \tag{6}$$

has to be investigated. Now $\gcd(tu, t^2 + u^2) = 1$. Therefore, (6) implies $tu \mid vw$. As we get similarly $vw \mid tu$, too, we infer

$$tu = vw \tag{7}$$

and (6) becomes

$$\frac{(v^2+t^2)(v^2+u^2)}{v^2} = t^2 + u^2 + v^2 + w^2 = kptu.$$

Therefore, $p$ has to divide either $v^2 + t^2$ or $v^2 + u^2$. In the case $p \equiv -1 \pmod 4$, i. e. when $-1$ is a quadratic non-residue mod $p$, this means that $p$ divides $v$ (and $t$ or $u$). But since the same argument is valid for $w$ instead of $v$, we have $p \mid v, w$ contradicting the coprimality of $v$ and $w$. Thus the infinitely many primes with $p \equiv -1 \pmod 4$ have no fantastic multiple and part (a) is solved.

For part (b) we choose $v = 1$ and substitute $w = tu$. Thus we are looking for integers $t$ and $u$ such that

$$1 + t^2 + u^2 + t^2u^2 = kptu.$$

Here we choose[1] $t = F_{2l+1}$, $u = F_{2l-1}$ and use the identity[2] $1 + F_{2l+1}^2 = F_{2l+3}F_{2l-1}$ to obtain

$$(1+t^2)(1+u^2) = (1+F_{2l+1}^2)(1+F_{2l-1}^2) = F_{2l+3}F_{2l-1}F_{2l+1}F_{2l-3} = kptu = kpF_{2l+1}F_{2l-1},$$

i. e. $F_{2l+3}F_{2l-3} = kp$. Therefore every prime factor of the Fibonacci number $F_{2l+3}$ has a fantastic multiple.

In view of the well-known formula $\gcd(F_a, F_b) = F_{\gcd(a,b)}$ it is clear that $F_a$ and $F_b$ are relatively prime, if $a$ and $b$ are different prime numbers. Hence we know that infinitely many prime numbers have a fantastic multiple, which solves part (b).

[1] It is a well-known problem that $tu \mid t^2 + u^2 + 1$ with $t > u$ is only possible if $t$ and $u$ are Fibonacci numbers of the form $t = F_{2l+1}$, $u = F_{2l-1}$ in which case $t^2 + u^2 + 1 = 3tu$.

[2] This is a special case of Vajda's identity $F_{n+i}F_{n+j} - F_nF_{n+i+j} = (-1)^n F_iF_j$