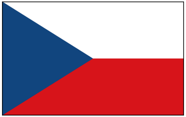
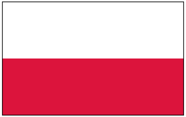


9th Czech-Polish-Slovak Junior Mathematical Competition

18.06.2021 — Individual Contest Solutions

Problem I-1.

We are given a table 2×2 with a positive integer written in each cell. If we add up the product of numbers in the first column, the product of numbers in the second column, the product of numbers in the first row, and the product of numbers in the second row, we will get 2021.

a) Determine the possible values of the sum of the four numbers in the table.

b) Find the number of tables satisfying the statement in which four entries are pairwise different.

Official solution

a) Denote numbers in the table by a, b, c, d, as shown in the picture.

a	c
b	d

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As $a + d \geqslant 2$ and $b + c \geqslant 2$, this means that $\{a + d, b + c\} = \{43, 47\}$ and in consequence $a + b + c + d = 90$.

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b) If $a + d = 43$ and $b + c = 47$, then the pair (a, d) can be chosen in 42 ways, and the pair (b, c) — in 46 ways.

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In order to give the final answer, we need to subtract the number of tables where some not all numbers are equal. Consider any such table.

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In order to give the final answer, we need to subtract the number of tables where some not all numbers are equal. Consider any such table. As 43 and 47 are odd, we have $a \neq c$ and $b \neq d$. Moreover $a + d \neq b + c$, so exactly one of the equalities $a = b$, $a = c$, $d = b$, $d = c$ holds.

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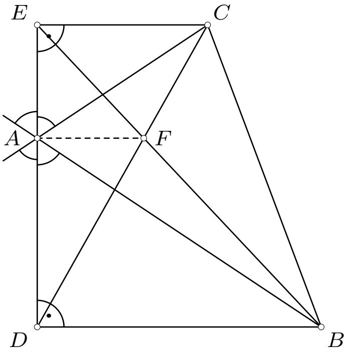
Ultimately, the answer is $2 \cdot 42 \cdot 46 - 8 \cdot 42 = 3528$.

Problem I-2.

Let ABC be an acute triangle. Denote by D and E the projections of B and C, respectively, on the external angle bisector of $\angle BAC$. Let F be the intersection point of BE and CD. Prove that $AF \perp DE$.

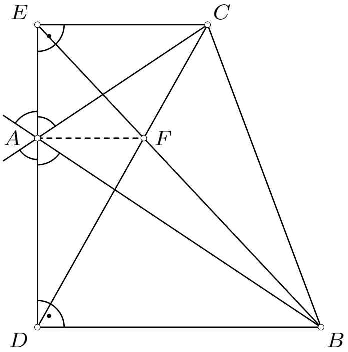
Official solution



Official solution

As $\angle BAD = \angle CAE$, right triangles ABD and ACE are similar and

$$\frac{AD}{AE} = \frac{BD}{CE}.$$



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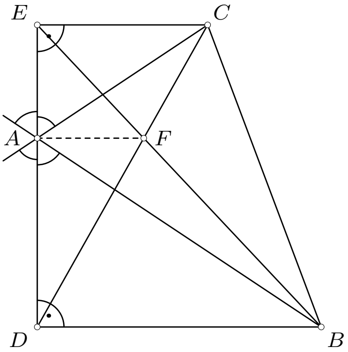
Moreover $BD \parallel CE$, so

$$\frac{BD}{CE} = \frac{FD}{FC}.$$



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$$\frac{AD}{AE} = \frac{BD}{CE}.$$

Moreover $BD \parallel CE$, so

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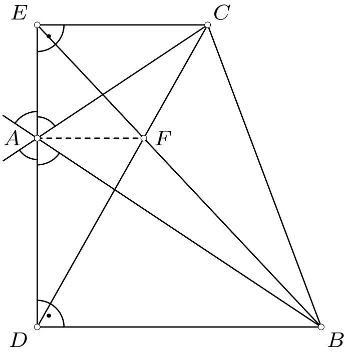
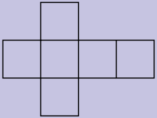
Combining the two equalities, we get that $AF \parallel CE$, so $AF \perp DE$.



Problem I-3.

The *cross* is a figure consisting of 6 unit squares presented in the picture below (and any other figure obtained from it by rotation).



Determine the largest number of crosses that can be cut from a 6×11 piece of paper divided into unit squares (each cross should consist of six such squares).

Official solution

Official solution

Let us cover boundary cells with 15 dominoes, as shown in the picture.

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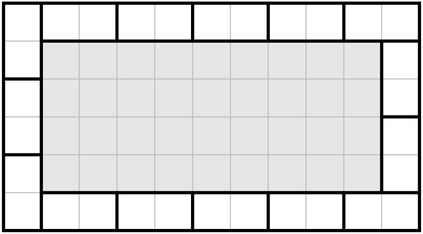
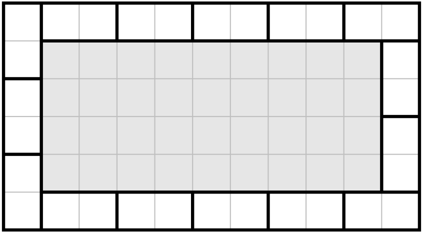


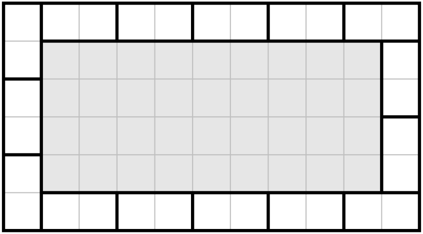
Official solution

Let us cover boundary cells with 15 dominoes, as shown in the picture. In each such domino there is at most one cell belonging to some cross.



Official solution

Let us cover boundary cells with 15 dominoes, as shown in the picture. In each such domino there is at most one cell belonging to some cross. Therefore, all crosses can together contain at most $6 \cdot 11 - 15 = 51$ cells, which means that there are at most 8 crosses.



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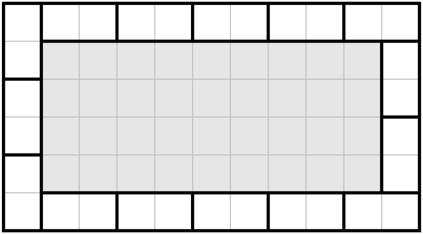
Let us cover boundary cells with 15 dominoes, as shown in the left picture. In each such domino there is at most one cell belonging to some cross. Therefore, all crosses can together contain at most $6 \cdot 11 - 15 = 51$ cells, which means that there are at most 8 crosses. Eight is actually possible (see the right picture).



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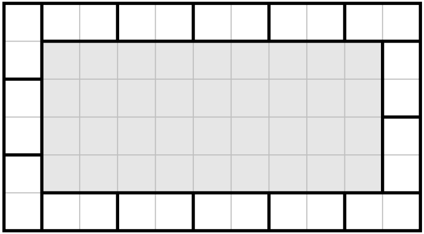

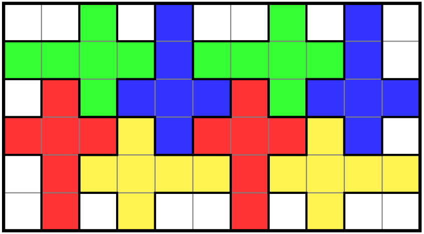

Problem I-4.

Determine the smallest value that an expression

$$x^4 + y^4 - x^2y - xy^2$$

attains, where x and y are positive real numbers satisfying $x + y \leqslant 1$.

Official solution

Note that

$$x^4 + y^4 - x^2y - xy^2$$

Official solution

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$$x^4 + y^4 - x^2 y - x y^2$$

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$$x^4 + y^4 - x^2y - xy^2 = \left(x^2 - y^2\right)^2 + 2x^2y^2 - xy(x + y)$$

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Official solution

Note that

$$x^4 + y^4 - x^2y - xy^2 = (x^2 - y^2)^2 + 2x^2y^2 - xy(x+y) \geqslant$$

$$\geqslant (x^2 - y^2)^2 + 2\left(xy - \frac{1}{4}\right)^2 - \frac{1}{8}$$

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Note that

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$$\geqslant \left(x^2 - y^2\right)^2 + 2\left(xy - \frac{1}{4}\right)^2 - \frac{1}{8} \geqslant -\frac{1}{8},$$

Official solution

Note that

$$x^4+y^4-x^2y-xy^2=\left(x^2-y^2\right)^2+2x^2y^2-xy(x+y)\geqslant$$

$$\geqslant\left(x^2-y^2\right)^2+2\left(xy-\frac{1}{4}\right)^2-\frac{1}{8}\geqslant-\frac{1}{8},$$

If $x=y=\frac{1}{2}$ then $-\frac{1}{8}$ is achieved.

Problem I-5.

Let $ABCDEFG$ be a regular 7-gon. Lines AB and CE intersect at P. Find $|\angle PDG|$.

Official solution

Official solution

Denote by Q the intersection point of segments DG and CE.

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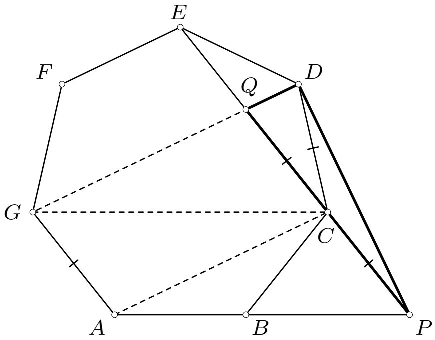

Official solution

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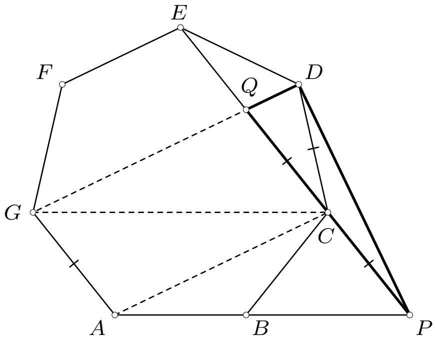

As the 7-gon is regular, we have $AB \parallel CG$, $AC \parallel DG$ and $AG \parallel CE$.

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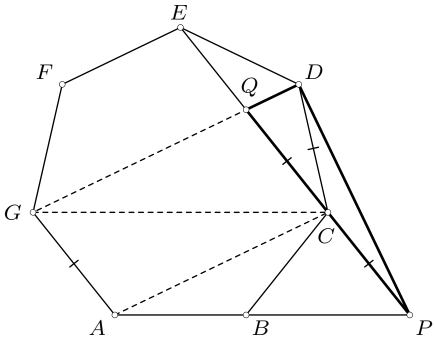

As the 7-gon is regular, we have $AB \parallel CG$, $AC \parallel DG$ and $AG \parallel CE$. Therefore, $APCG$ and $ACQG$ are parallelograms hence $CP = CQ = AG = CD$.

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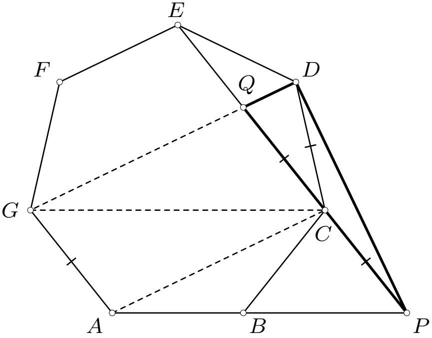

As the 7-gon is regular, we have $AB \parallel CG$, $AC \parallel DG$ and $AG \parallel CE$. Therefore, $APCG$ and $ACQG$ are parallelograms hence $CP = CQ = AG = CD$. This means that triangle DPQ is right and $\angle PDG = 90^\circ$.