# CPSA 2021 – solutions

**1.** Find all quadruples $(a, b, c, d)$ of positive integers satisfying $\gcd(a, b, c, d) = 1$ and

$$a \mid b + c, \quad b \mid c + d, \quad c \mid d + a, \quad d \mid a + b.$$

*Vítězslav Kala (Czech Republic)*

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**Solution.** Without loss of generality, assume that $a = \max\{a, b, c, d\}$. Then

$$a \leq b + c \leq 2a,$$

and so we have 2 possible cases:

CASE 1. $b + c = 2a$.

In this case, $a = b = c$, and so

$$a = c \mid (d + a) - a = d.$$

But $a \geq d$, and so we must have $a = b = c = d$, and by the coprimality assumption, we get the solution $(1, 1, 1, 1)$.

CASE 2. $b + c = a$.

Let $c + d = kb$ for some positive integer $k$. The relation $c \mid d + a$ then implies

$$c \mid c + d + (b + c) = (k + 1)b + c,$$

and so $(k + 1)b = lc$ for some positive integer $l$.

Moreover, we have

$$d \mid a + b = 2b + c,$$

and so $md = m(kb - c) = 2b + c$ for some positive integer $m$.

We now have the system

$$(k + 1)b = lc$$
$$(km - 2)b = (m + 1)c.$$

From the second equation, we see that $km - 2 > 0$, and so

$$l = (k + 1)\frac{b}{c} = \frac{(k + 1)(m + 1)}{km - 2} = 1 + \frac{k + m + 3}{km - 2}.$$

Since $l$ is an integer, we have

$$km - 2 \mid k + m + 3.$$

However, if $(k - 1)(m - 1) > 6$, then $km - 2 > k + m + 3$, which would contradict $km - 2 \mid k + m + 3$.

Note that $m = 1$ is not possible, for then we would have $d = 2b+c > b+c = a$. Therefore, there are 5 remaining cases.

CASE 2a. $k = 1$.

Then $m - 2 \mid m + 4$, and so $m - 2 \mid 6$, i.e. $m$ is one of the numbers $3, 4, 5, 8$ (recall that we know that $m - 2 = km - 2 > 0$). The corresponding values of $l$ are then $8, 5, 4, 3$. By the coprimality assumption, this uniquely determines the solutions as

$$(a, b, c, d) \in \{(5, 4, 1, 3), (7, 5, 2, 3), (3, 2, 1, 1), (5, 3, 2, 1)\}.$$

CASE 2b. $k = 2$ for $2 \le m \le 7$.

We then have $2m - 2 \mid m + 5$, and so the only possibilities are $(m, l) \in \{(3, 3), (7, 2)\}$, to which the corresponding solutions are

$$(a, b, c, d) \in \{(2, 1, 1, 1), (5, 2, 3, 1)\}.$$

CASE 2c. $m = 2$ for $3 \le k \le 7$.

Then $2k - 2 \mid k + 5$, and so the only possibilities are $(k, l) \in \{(3, 3), (7, 2)\}$ and so

$$(a, b, c, d) \in \{(7, 3, 4, 5), (5, 1, 4, 3)\}.$$

CASE 2d. $k = 3$ for $3 \le m \le 4$.

Then $3m - 2 \mid m + 6$. The only possibility is $m = 4$, and so $l = 2$ and

$$(a, b, c, d) = (3, 1, 2, 1).$$

CASE 2e. $k = 4$ and $m = 3$.

Here, we get $l = 2$ and

$$(a, b, c, d) = (7, 2, 5, 3).$$

Altogether, we have found the following solutions:

$$(a, b, c, d) \in \{(7, 5, 2, 3), (7, 3, 4, 5), (7, 2, 5, 3), (5, 4, 1, 3), (5, 3, 2, 1),$$
$$(5, 2, 3, 1), (5, 1, 4, 3), (3, 2, 1, 1), (3, 1, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1)\}$$

(and all their rotations when we remove the assumption that $a$ is largest).

**Alternative solution.** We will show a different finish of case 2 from the previous solution. we need to fulfill

$$b \mid c + d, \qquad c \mid d + b, \qquad d \mid 2b + c.$$

We will distinguish cases based on which of the numbers $b$, $c$, $d$ is largest.

CASE 2a. $b = \max(b, c, d)$.

Then $b = (c+d)/2$ or $b = c+d$ by the same reasoning as in in case 1. The first statement yields $b = c = d$, which gives $(a, b, c, d) = (2, 1, 1, 1)$ when combined with the coprimality condition.

If, on the other hand, $b = c + d$ holds, then the conditions above ensure $c \mid 2d$ and $d \mid 3c$, thus $c \mid 2d \mid 6c$, so that $2d \in \{c, 2c, 3c, 6c\}$. Each of these possibilities determines $a$ and $b$ uniquely by the coprimality condition. We arrive at the solutions

$$(a, b, c, d) \in \{(3, 2, 1, 1), (5, 3, 2, 1), (5, 4, 1, 3), (7, 5, 2, 3)\}.$$

CASE 2b. $c = \max(b, c, d)$.

By the same reasoning as in case 2a, we get

$$(a, b, c, d) \in \{(2, 1, 1, 1), (5, 2, 3, 1), (5, 1, 4, 3), (7, 2, 5, 3), (3, 1, 2, 1)\}.$$

CASE 2c. $d = \max(b, c, d)$.

Because of $b + c = a \geq d \geq (2b + c)/3$, we can only have $d \mid 2b + c$ for $d = (2b + c)/2$ or $d = (2b + c)/3$. Again, the latter case yields $b = c = d$ and the solution $(2, 1, 1, 1)$.

For $d = (2b + c)/2 = b + c/2$, we find that $c$ has to be even, and so $c = 2C$ for a positive integer $C$. Now, we obtain $b \mid c + d = 2C + (b + C)$, which means $b \mid 3C$, as well as $C \mid c \mid d + b = (b + C) + b$, and therefore $C \mid 2b$. We infer $b \mid 3C \mid 6b$ and from that $3C \in \{b, 2b, 3b, 6b\}$. Only $3C = 2b$ yields a new solution, specifically $(7, 3, 4, 5)$.

**2.** In an acute triangle $ABC$, the incircle $\omega$ touches $BC$ at $D$. Let $I_a$ be the excenter of $ABC$ opposite to $A$, and let $M$ be the midpoint of $DI_a$. Prove that the circumcircle of triangle $BMC$ is tangent to $\omega$. *Patrik Bak (Slovakia)*

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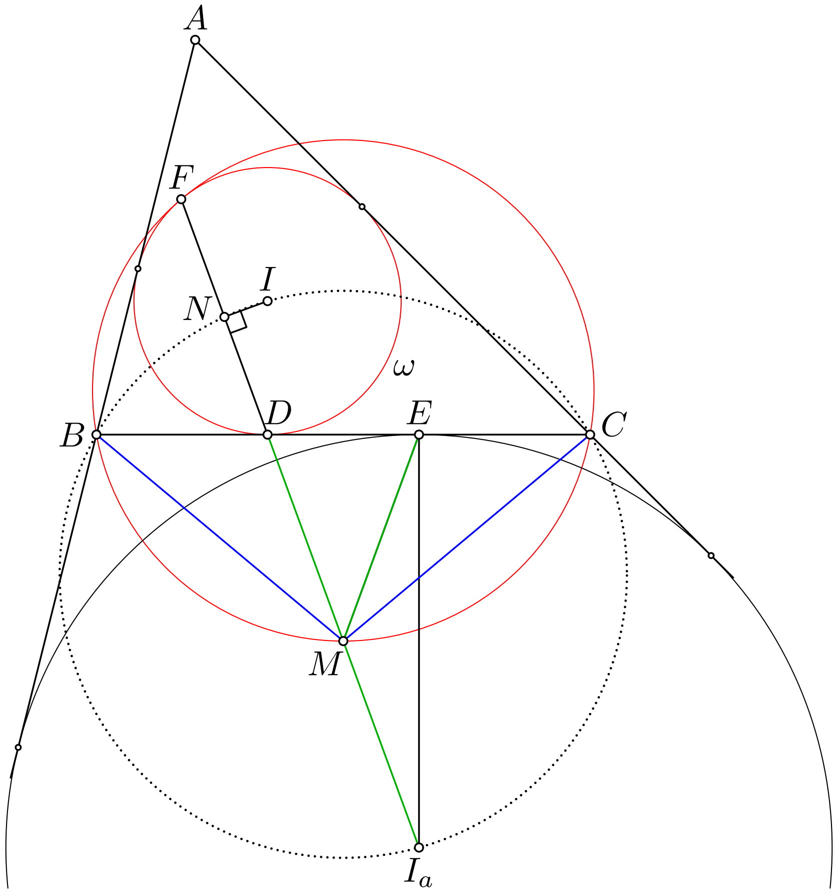
**Solution.** Let $I$ be the incenter of $ABC$ and let $F$ be the second intersection point of $MD$ and $\omega$ and let $N$ be the midpoint of $FD$. Points $B, N, I, C, I_a$ are concyclic, as they lie on the circle with a diameter $II_a$. The power of a point gives

$$DF \cdot DM = 2DN \cdot \frac{1}{2}DI_a = DB \cdot DC,$$

which means that $F, B, M, C$ are concyclic.

Let $E$ the projection of $I_a$ on $BC$. It is well-known that $D$ and $E$ are symmetric with respect to the midpoint of $BC$. Since $MD = MI_a = ME$, the congruence of triangles $MDB$ and $MEC$ gives $MB = MC$.

It remains to realize that the circle through $F$, $B$, $M$, and $C$ is tangent to $\omega$ at $F$. This can be seen from homothety: Without loss of generality let $BC$ be horizontal. Then $D$ is a lowest point of $\omega$, Since $MB = MC$, also $M$ is the lowest point of the circumricle of $MBC$. If there is a circle through $M, B, C$ tangent to $\omega$, then the tangency point




must be the second intersection point of the circumcircle of $MBC$ and line $MD$, which is indeed $F$.

**3.** For any two convex polygons $P_1$ and $P_2$ with mutually distinct vertices, denote by $f(P_1, P_2)$ the total number of their vertices that lie on a side of the other polygon. For each positive integer $n \geq 4$, determine

$$\max\{f(P_1, P_2) \mid P_1 \text{ and } P_2 \text{ are convex } n\text{-gons}\}.$$

(We say that a polygon is *convex* if all its internal angles are strictly less than 180°.)

*Josef Tkadlec (Czech Republic)*

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**Solution.** We will show that $F(n) = \lfloor 4n/3 \rfloor$ for any $n \geq 4$.

For the construction, see Figure 1.

For the bound, fix $n \geq 4$ and two convex $n$-gons $P_1$, $P_2$. Call any of the $2n$ vertices *good* if it lies on a side of the other polygon.

If the interiors of $P_1$ and $P_2$ do not intersect, then at most 2 points are good. Indeed, in this case, there is a line $\ell$ such that each corresponding (closed) half-plane contains one

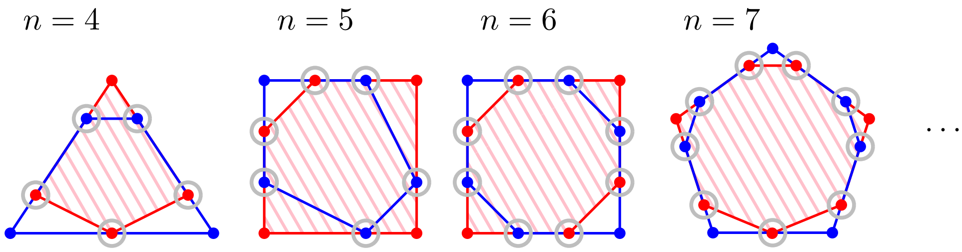


Figure 1

of the polygons. Since both the polygons are convex, at most $2 + 2$ of their vertices lie on $\ell$, hence $f(P_1, P_2) \leq 2 < \lfloor 4n/3 \rfloor$ for any $n \geq 4$.

Suppose that the interiors do intersect and take any point $X$ inside both $P_1$ and $P_2$, not lying on any line through two vertices. A rotating ray emanating from $X$ defines a cyclical order of the $2n$ vertices. It suffices to show that among any three consecutive vertices in this order, at most 2 are good – the bound then follows by summing over all consecutive triples.

Color vertices of $P_1$ black (B) and vertices of $P_2$ white (W). By symmetry, it suffices to distinguish three cases of the colors of the three consecutive vertices: WWW, WWB, and WBW.

Split the plane into $n$ "wedges" with a shared apex $X$ and rays passing through all (black) vertices of $P_1$. Note that since $X$ is inside $P_1$, these wedges are convex and each wedge contains precisely one side of $P_1$ (and each side of $P_1$ is contained in precisely one wedge).

In each of the three cases, we suppose that all three vertices are good, argue that the three vertices in fact lie on the same line and then reach a contradiction with the convexity of $P_2$ by finding three collinear white vertices on that line (see Figure 2).

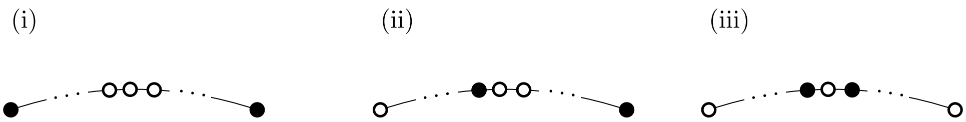


Figure 2

(i) WWW: All three white vertices lie in the same wedge, hence on the same side of $P_1$, a contradiction.

(ii) BWW: Both white vertices lie in the same wedge, hence on the same side of $P_1$. This side has the black vertex as one endpoint, hence the three vertices are collinear. Since the black vertex is also good, there is a third white vertex on that line on the other side of the black vertex, a contradiction.

(iii) BWB: The white vertex lies on the segment connecting the two black vertices, hence the three vertices are collinear. Since both the black vertices are also good, there is one more white vertex on each side, a contradiction.

This completes the proof.

*Remark.* One can show that $F(3) = 3 \neq \lfloor 4 \cdot 3/3 \rfloor$.

**4.** Determine the number of 2021-tuples of positive integers such that the number 3 is an element of the tuple and consecutive elements of the tuple differ by at most 1.

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**Solution.** First, we count the number of such tuples ignoring the first property.

Any tuple $(a_1, \dots, a_{2021})$ having the second property is uniquely determined by $\min_{i=1}^{2021} a_i$ and the tuple $(a_2 - a_1, \dots, a_{2021} - a_{2020}) \in \{-1, 0, 1\}^{2020}$.

Hence, if the minimum is given, there are $3^{2020}$ tuples satisfying only the second condition.

To account for the first condition, that is, to have 3 as an entry of the sequence, we need the minimum from above to belong to $\{1, 2, 3\}$ on one hand, and the maximum of all $a_i$ to be greater than or equal to 3 on the other hand. This is equivalent to $\min_{i=1}^{2021} a_i \in \{1, 2, 3\}$ and for the sequence $(a_1, \dots, a_{2021})$ to have entries from $\{1, 2\}$ (these sequences all have the second property).

Therefore, the desired number of sequences fulfilling both conditions is $3 \cdot 3^{2020} - 2^{2021} = 3^{2021} - 2^{2021}$.

**5.** The sequence $a_1, a_2, a_3, \dots$ satisfies $a_1 = 1$, and for all $n \geq 2$, it holds that

$$a_n = \begin{cases} a_{n-1} + 3 & \text{if } n - 1 \in \{a_1, a_2, \dots, a_{n-1}\}; \\ a_{n-1} + 2 & \text{otherwise.} \end{cases}$$

Prove that for all positive integers $n$, we have

$$a_n < n \cdot (1 + \sqrt{2}).$$

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**Solution.** First, it is easy to see that for any $n \geq 2$, we have $a_n = 2n + k - 1$ where $k = \max\{i : a_i < n\}$. Indeed, $a_n$ is obtained by adding to $a_1$ twos and threes in $n - 1$ steps, where 3 is added in steps $a_1, a_2, \dots, a_k$, and two is added in the remaining $n - 1 - k$ steps. Hence $a_n = a_1 + 3k + 2(n - k - 1) = 2n + k - 1$. Also, note that such a $k$ satisfies $k + 1 < n$ provided $n \geq 3$.

Now, we shall prove the following stronger statement: For any $n \geq 1$, we have

$$(1 + \sqrt{2})n - 2 < a_n < (1 + \sqrt{2})n.$$

This is clearly true for $n = 1$ and $n = 2$. For the inductive step, let $n \geq 3$ and suppose that this holds for all indices smaller than $n$. Write $a_n = 2n + k - 1$ where $k = \max\{i\colon a_i < n\}$, so that we have $a_k < n \leq a_{k+1}$. We have to prove that

$$(1+\sqrt{2})n - 2 < 2n + k - 1 < (1+\sqrt{2})n.$$

This is equivalent to

$$(\sqrt{2}-1)n - 1 < k < (\sqrt{2}-1)n + 1.$$

We have

$$(\sqrt{2}-1)n - 1 \leq (\sqrt{2}-1)a_{k+1} - 1 < (\sqrt{2}-1)(\sqrt{2}+1)(k+1) - 1 = k,$$

where the first inequality holds since $n \leq a_{k+1}$, and the second one holds by the induction hypothesis applied to $k+1$.

Similarly,

$$\begin{aligned}(\sqrt{2}-1)n + 1 > (\sqrt{2}-1)a_k + 1 &> (\sqrt{2}-1)((\sqrt{2}+1)k - 2) + 1 \\ &= k - 2(\sqrt{2}-1) + 1 = k + 3 - 2\sqrt{2} > k\end{aligned}$$

because $n > a_k$ and, by the inductive hypothesis, $a_k > (1+\sqrt{2})k - 2$.

This finishes the proof of the double inequality

$$(1+\sqrt{2})n - 2 < a_n < (1+\sqrt{2})n,$$

and we are done.

**6.** Let $ABC$ be an acute triangle and suppose points $A$, $A_b$, $B_a$, $B$, $B_c$, $C_b$, $C$, $C_a$, and $A_c$ lie on its perimeter in this order. Let $A_1 \neq A$ be the second intersection point of the circumcircles of triangles $AA_bC_a$ and $AA_cB_a$. Analogously, $B_1 \neq B$ is the second intersection point of the circumcircles of triangles $BB_cA_b$ and $BB_aC_b$, and $C_1 \neq C$ is the second intersection point of the circumcircles of triangles $CC_aB_c$ and $CC_bA_c$. Suppose that the points $A_1$, $B_1$, and $C_1$ are all distinct, lie inside the triangle $ABC$, and do not lie on a single line. Prove that lines $AA_1$, $BB_1$, $CC_1$, and the circumcircle of triangle $A_1B_1C_1$ all pass through a common point.

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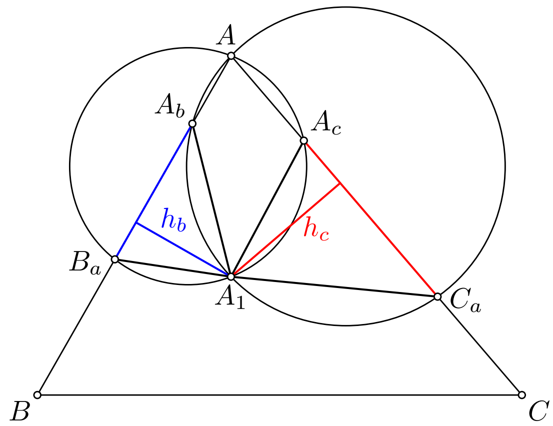
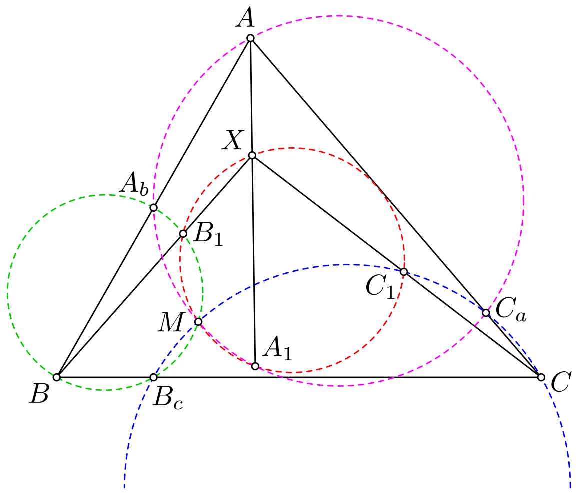
**Solution.** First, we prove will that the three lines are concurrent.

Point $A_1$ is the center of the spiral similarity that maps $B_aA_b$ to $A_cC_a$, and so the triangles $A_1B_aA_b$ and $A_1A_cC_a$ are similar (this is also easy to verify by direct angle-chasing). We aim to use the trigonometric form Ceva's Theorem. Let $h_c$ and $h_b$ be the distances of $A_1$ to the sides $AB$ and $AC$, respectively. Using the similar triangles, we get

$$\frac{\sin \sphericalangle BAA_1}{\sin \sphericalangle A_1AC} = \frac{h_c/AA_1}{h_b/AA_1} = \frac{h_c}{h_b} = \frac{B_aA_b}{A_cC_a},$$




which is cyclic in terms of $A$, $B$, $C$, hence the lines are concurrent by the trigonometric form Ceva's Theorem

Next, we will prove that the intersection point of lines $AA_1$, $BB_1$ and $CC_1$ lines on the circumcircle of $A_1B_1C_1$.

Focus on points $B_1$, $B_c$ and $C_1$ lying on the lines determined by the vertices of $XBC$. Due to Miquel's theorem, the circumcircles of $XB_1C_1$, $BB_1B_c$ and $CC_1B_c$ are concurrent, denote their common point by $M$.

Applying Miquel's theorem on points $A_b$, $B_c$, and $C_a$ lying on the sides of $ABC$ gives that $M$ lies also on the circumcircle of $AA_bC_a$. Due to this, we can repeat the logic used to define $M$ with regards to triangle $XAB$ to prove that $X$, $A_1$, $B_1$ and $M$ are concyclic. Altogether, points $X$, $A_1$, $B_1$, $C_1$ and $M$ are concyclic, so we are done.