TENTH MIDDLE EUROPEAN MATHEMATICAL OLYMPIAD

VÖCKLABRUCK

AUSTRIA 2016

Contest Problems with Solutions

Jury & Problem Selection Committee

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The Note of Confidentiality **IMPORTANT**

We would like to remind everybody of the following MEMO regulation:

These exam problems
have to be kept strictly confidential
until the contest will have been finished.

Jury & Problem Selection Committee

selected

12 problems submitted by the following countries:

	T-1 Croatia
	T-2 Lithuania
I-1 Austria	T-3 Croatia
I-2 Switzerland	T-4 Austria
I-3 Slovakia	T-5 Croatia
I-4 Croatia	T-6 Poland
	T-7 Slovakia
	T-8 Austria

The Problem Selection Committee would also like to thank **Roger Labahn** for providing the LaTeX templates.

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I-1 A

Let $n \geqslant 2$ be an integer and x_1, x_2, ..., x_n be real numbers satisfying

(a) $x_j > -1$ for $j = 1, 2, \ldots, n$ and

(b) $x_1 + x_2 + \cdots + x_n = n$.

Prove the inequality

$$\sum_{j=1}^{n} \frac{1}{1+x_j} \geqslant \sum_{j=1}^{n} \frac{x_j}{1+x_j^2}$$

and determine when equality holds.

Solution. We have to prove

$$\sum_{j=1}^{n} \frac{1}{1+x_j} - \sum_{j=1}^{n} \frac{x_j}{1+x_j^2} = \sum_{j=1}^{n} \frac{1-x_j}{(1+x_j)(1+x_j^2)} \geqslant 0.$$

We use the supporting line method and consider the function f defined by

$$f(x) = \frac{1-x}{(1+x)(1+x^2)}$$

for all $x > -1$. The tangent line of f at $x = 1$ is given by $y = \frac{1-x}{4}$. We claim that

$$f(x) = \frac{1-x}{(1+x)(1+x^2)} \geqslant \frac{1-x}{4}$$

for all $x > -1$ with equality for $x = 1$. For $x \geqslant 1$ we get $4 \leqslant (1+x)(1+x^2)$ and for $-1 < x \leqslant 1$ we get $4 \geqslant (1+x)(1+x^2)$. Both inequalities are obviously true.

Now we conclude that

$$\sum_{j=1}^{n} \frac{1-x_j}{(1+x_j)(1+x_j^2)} \geqslant \sum_{j=1}^{n} \frac{1-x_j}{4} = 0,$$

and we are done.

Equality occurs if and only if all n numbers are equal to 1.

Solution. Since $1 + x_j > 0$ for $j = 1, 2, \ldots, n$, Cauchy-Schwarz inequality yields

$$\sum_{j=1}^{n} \frac{1}{1+x_j} \cdot \sum_{j=1}^{n} (1+x_j) \geqslant \left(\sum_{j=1}^{n} 1 \right)^2,$$

which is equivalent to

$$\sum_{j=1}^{n} \frac{1}{1+x_j} \geqslant \frac{n}{2} \ .$$

It therefore suffices to prove that

$$\sum_{j=1}^{n} \frac{x_j}{1+x_j^2} \leqslant \frac{n}{2} \ ,$$

but this last inequality is equivalent to the trivial one

$$\sum_{j=1}^{n} \frac{(1-x_j)^2}{1+x_j^2} \geqslant 0 \ ,$$

so the inequation is proven.

In the last equation, we have equality if and only if $x_j = 1$ for $j = 1, 2, \ldots, n$, and one can easily see that this is indeed a case of equality, so it is the only case of equality.

Solution The inequality is equivalent to

$$\sum_{i=1}^{n} \frac{1-x_i}{(1+x_i)(1+x_i^2)} \geqslant 0$$

As the functions $f(x) = 1 - x$ and $g(x) = \frac{1}{(1+x)(1+x^2)}$ are both strictly decreasing, we can apply the Chebychev inequality to obtain:

$$n \cdot \sum_{i=1}^{n} \frac{1-x_i}{(1+x_i)(1+x_i^2)} \geqslant \left(\sum_{i=1}^{n} 1 - x_i \right) \left(\sum_{i=1}^{n} \frac{1}{(1+x_i)(1+x_i^2)} \right) = 0$$

So we're done.

Solution (via Lagrange multipliers) Let us write $f(x) = \dfrac{1}{1+x} - \dfrac{x}{1+x^2}$. We want to show that the expression $f(x_1) + \ldots + f(x_n)$ in the domain

$$D : x_1, \ldots, x_n > -1, \ x_1 + \ldots + x_n = n$$

attains its minimal value 0 exactly at the point $x_1 = \ldots = x_n = 1$.

We first consider the boundary of D. This means that w. l. o. g. we may assume that $x_1 = -1$, in which case the expression attains the value $+\infty$, which is not the minimum.

We now look for minima in the interior of the domain: The method of Lagrange multipliers gives the Langrange function

$$F(x_1, \ldots, x_n, \lambda) = \sum_{j=1}^{n} \left(f(x_j) - \lambda(x_j - 1) \right)$$

and results in the system of equations

$$\begin{aligned} f'(x_j) &= \lambda, \qquad j = 1, \ldots, n, \\ \sum_{j=1}^{n} x_j &= n. \end{aligned}$$

Now note that

- $f'(x) = \dfrac{-1}{(1+x)^2} + \dfrac{x^2-1}{(1+x^2)^2} = \dfrac{2(x^3-x^2-x-1)}{(1+x)^2(1+x^2)^2}$.
- $f'(x) < 0$ for $-1 < x \leqslant 1$. This is obvious from the first expression for $f'(x)$.
- $f'(x) > 0$ for $x > 2$. This can be seen from the second expression for $f'(x)$, since for $x > 2$ we have $1 + x + x^2 < \dfrac{x^3}{8} + \dfrac{x^3}{4} + \dfrac{x^3}{2} < x^3$.
- $f''(x) = \dfrac{2x(3-x^2)}{(1+x^2)^3} + \dfrac{2}{(1+x)^3}$.
- $f''(x) > 0$ for $-1 < x < 2$ can be shown by considering the following three cases:
  - For $-1 < x < 0$, we have

    $$\frac{1}{(1+x)^3} = \frac{1}{(1+x)(1+2x+x^2)} > \frac{1}{(1+x)(1+x^2)}.$$

    Thus we get

    $$\begin{aligned} f''(x) &> \frac{2x(3-x^2)}{(1+x^2)^3} + \frac{2}{(1+x)(1+x^2)} = \frac{2x(3-x^2)(1+x) + 2(1+x^2)^2}{(1+x)(1+x^2)^3} = \\ &= \frac{(6x+6x^2-2x^3-2x^4) + (2+4x^2+2x^4)}{(1+x)(1+x^2)^3} = \frac{2+6x+10x^2-2x^3}{(1+x)(1+x^2)^3} = \\ &= \frac{\frac{1}{2}(2+3x)^2 + \frac{11}{2}x^2 - 2x^3}{(1+x)(1+x^2)^3} > 0. \end{aligned}$$

  - For $0 \leqslant x \leqslant \sqrt{3}$, the assertion is obvious.
  - For $\sqrt{3} < x < 2$, the assertion follows from

    $$\frac{2}{(1+x)^3} > \frac{2}{27}, \qquad \frac{2x(3-x^2)}{(1+x^2)^3} = -\frac{2x(x^2-3)}{(1+x^2)^3} > -\frac{2\cdot 2(2^2-3)}{(1+\sqrt{3}^2)^3} = -\frac{1}{16} > -\frac{2}{27}.$$

- Hence f' is strictly increasing for $-1 < x \leqslant 2$.

Since $\overline{x} = \frac{1}{n}\sum_{j=1}^{n} x_j = 1$, we know that $x_j \leqslant 1$ for some j. Therefore $\lambda < 0$. This means $x_j \leqslant 2$ for every $j = 1, \ldots, n$. From the monotonicity of f' in the domain $-1 < x \leqslant 2$, we now conclude $x_1 = \ldots = x_n$. In view of the condition $x_1 + \ldots + x_n = n$ this means $x_1 = \ldots = x_n = 1$. Since $f(1) = 0$, we are done.

I-2 C

There are $n \geqslant 3$ positive integers written on a blackboard. A move consists of choosing three numbers a, b, c on the blackboard such that they are the sides of a non-degenerate non-equilateral triangle and replacing them by $a+b-c$, $b+c-a$ and $c+a-b$.

Show that an infinite sequence of moves cannot exist.

Solution 1. We will show that the product of all the numbers on the blackboard can never increase. Indeed, for the three numbers a, b and c we have the inequalities

$$\begin{aligned} a^2 &\geqslant a^2-(b-c)^2 = (a+b-c)(a+c-b) \\ b^2 &\geqslant b^2-(a-c)^2 = (b+a-c)(b+c-a) \\ c^2 &\geqslant c^2-(a-b)^2 = (c+a-b)(c+b-a) \end{aligned}$$

Since by the conditions of the problem all factors on both sides are positive, we can multiply these equations and obtain

$$a^2b^2c^2 \geqslant (a+b-c)^2(b+c-a)^2(c+a-b)^2,$$

which is equivalent to

$$abc \geqslant (a+b-c)(b+c-a)(c+a-b). \tag{1}$$

Since every non-increasing sequence of positive integers is eventually constant, we see that the product of the numbers on the blackboard cannot change after a finite number of moves. Furthermore, it is clear that we have equality in (1) if and only if $a=b=c$, in which case the numbers on the blackboard do not change. It is now clear that after a finite number of moves the numbers on the blackboard will not change.

Solution 2. Since $(a+b-c)+(b+c-a)+(a+c-b)=a+b+c$, it is clear that the sum of the numbers on the blackboard is invariant. On the other hand, we learn from

$$(a+b-c)^2+(b+c-a)^2+(c+a-b)^2 = a^2+b^2+c^2+(a-b)^2+(a-c)^2+(b-c)^2$$

that the sum of squares of the numbers increases in every move, except in the case $a=b=c$, when nothing at all changes. But in view of the inequality $\sum x_i^2 \leqslant \left(\sum x_i\right)^2$, it now becomes evident that the number of possible "moves with effect" is bounded by the constant right-hand side.

Solution 3. Let m be the smallest integer on the blackboard and k the number of times that m is written on the blackboard. The number n of numbers on the blackboard will stay fixed throughout the process, while m and k might change.

We first prove that any move involving a number m will either decrease m or keeping m fixed and increasing k or change nothing on the blackboard:

- If $m = a \leqslant b < c$, we have $a + b - c < a = m$, so the new minimum is clearly smaller than the original m.
- If $m = a < b = c$, we have $a + b - c = c + a - b = a = m$, so the number k has increased while m is fixed.
- If $m = a = b = c$, nothing changes on the blackboard.

Now, if there are only finitely many moves that change anything and involve a minimal number then we can conclude by induction on n that we can only make finitely many moves that change anything. The most convenient base case is $n = 2$ where nothing changes because no three numbers can be chosen.

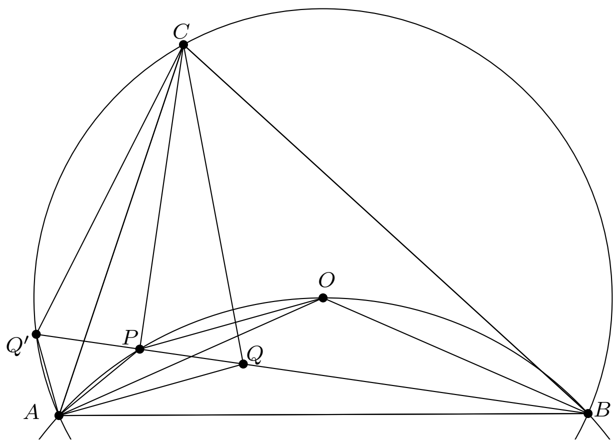
However, if there are infinitely many moves involving a minimal number that change something then we have seen above that either m decreases or m is fixed and k increases. Since k is at most n and m is at least 1 this is impossible, so the process has to terminate as desired.

(The same argument works with the maximum because it is bounded by the constant sum of the written integers.)

I-3 G

Let ABC be an acute-angled triangle with $\measuredangle BAC > 45^\circ$ and with circumcentre O. The point P lies in its interior such that the points A, P, O, B lie on a circle and BP is perpendicular to CP. The point Q lies on the segment BP such that AQ is parallel to PO.

Prove that $\measuredangle QCB = \measuredangle PCO$.



Solution 1. Since $\measuredangle BAC > 45^\circ$, we have $\measuredangle BOC > 90^\circ$, and the points A, P, O, B lie on a circle in this order.

Instead of the equality $\measuredangle QCB = \measuredangle PCO$, we show the equivalent statement $\measuredangle OCB = \measuredangle PCQ$.

We know that $\measuredangle OCB = 90^\circ - \measuredangle BAC$ and $\measuredangle PCQ = 90^\circ - \measuredangle CQP$. So we just need to show $\measuredangle BAC = \measuredangle CQP$. By the construction of P and Q, we have

$$\measuredangle PQA = \measuredangle QPO = \measuredangle BPO = \measuredangle BAO = \measuredangle OBA = 180^\circ - \measuredangle APO = \measuredangle QAP,$$

so the triangle AQP is isosceles with $PA = PQ$. Thus if Q' denotes the point obtained by reflecting Q about P, then $\measuredangle QAQ' = 90^\circ$.

Moreover, reusing some part of the above calculation and exploiting that O is the circumcentre of ABC, we find

$$\measuredangle AQ'B = \measuredangle AQ'Q = 90^\circ - \measuredangle Q'QA = 90^\circ - \measuredangle PQA = 90^\circ - \measuredangle OBA = \measuredangle ACB,$$

which proves that the quadrilateral $ABCQ'$ is cyclic. Finally, since $\measuredangle QPC = 90^\circ$ and $PQ = PQ'$, the triangle $Q'QC$ is isosceles with $CQ = CQ'$, whence

$$\measuredangle CQP = \measuredangle CQQ' = \measuredangle QQ'C = \measuredangle BQ'C = \measuredangle BAC,$$

as we wanted to show.

Another solution of the second part:

Let Q' be the point of intersection of BP with the circumcircle of the traingle ABC. Then we have

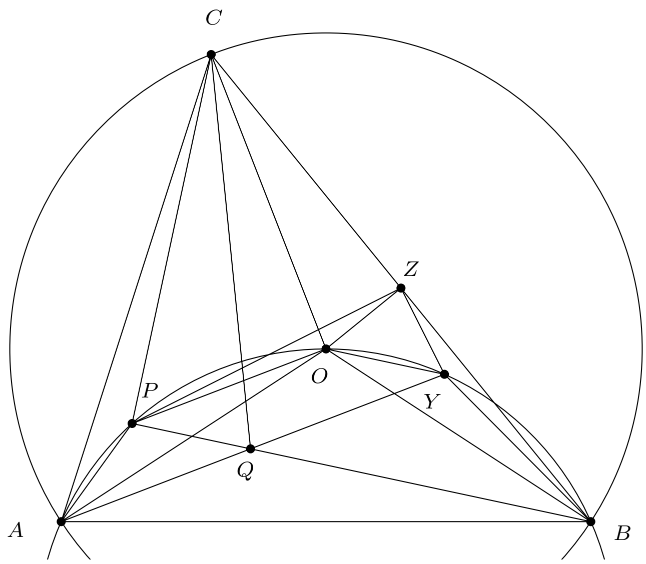
$$\measuredangle BQ'C = \measuredangle BAC = \alpha \quad \text{and} \quad \measuredangle AQ'B = \measuredangle ACB = \gamma.$$

Since $\measuredangle Q'PA = 180^\circ - \measuredangle APB = 180^\circ - 2\gamma$ we get $\measuredangle PAQ' = 180^\circ - \measuredangle AQ'B - \measuredangle Q'PA = \gamma$ and triangle APQ' is isosceles with $PQ' = PA$. Therefore we get $PQ = PA = PQ'$ and the point P is the midpoint of QQ'.

Finally, since $\measuredangle QPC = 90^\circ$ and $PQ = PQ'$, the triangle $Q'QC$ is isosceles with $CQ = CQ'$. Therefore

$$\measuredangle CQP = \measuredangle QQ'C = \measuredangle BAC,$$

as we wanted to show.



Solution 2. Since $\measuredangle BAC > 45^\circ$, we have $\measuredangle BOC > 90^\circ$, and the points A, P, O, B lie on a circle in this order.

Instead of showing the equality $\measuredangle QCB = \measuredangle PCO$, we prove the equivalent statement $\measuredangle OCB = \measuredangle PCQ$.

Let Y be the point of intersection of AQ with the circle through A, O and B. Due to $PO \parallel AY$ the quadrilateral $AYOP$ is an isosceles trapezoid with $PA = OY$. Since $AO = OB$, we can conclude $OP = YB$, e.g. by proving the congruence of the triangles AOP and OBY. Therefore, $PBYO$ is an isosceles trapezoid as well and we get that $PQYO$ is a parallelogram.

The triangle AQP is isosceles due to $PA = OY = PQ$ and, in addition, similar to triangle ABO because of $\measuredangle APB = \measuredangle AOB$.

Now let Z be the midpoint of BC. Since $\measuredangle BPC = 90^\circ$, we know that Z is the center point of the circle through P, B and C. Hence we get $ZC = ZP = ZB$. Since triangle PBZ is isosceles, we have $\measuredangle ZBP = \measuredangle BPZ$. Because of $\measuredangle ZBP = \measuredangle ZBY + \measuredangle YBP$ and $\measuredangle BPZ = \measuredangle BPO + \measuredangle OPZ$ and $\measuredangle YBP = \measuredangle BPO$, we conclude $\measuredangle ZBY = \measuredangle OPZ$. Now we have the equality of two corresponding sides and the enclosed angle. Therefore, the triangles YBZ and POZ are congruent, yielding $OZ = YZ$.

Since $\measuredangle OZY = 90^\circ - \measuredangle YZB = 90^\circ - \measuredangle PZO = \measuredangle CZP$ and $ZC = ZP$, we see that the triangles OYZ and CPZ are similar. Therefore, we get

$$\frac{CP}{OY} = \frac{CZ}{OZ}, \quad \text{hence} \quad \frac{CP}{PQ} = \frac{CZ}{OZ}.$$

Now the similarity of the triangles PQC and OZC follows from $\measuredangle QPC = \measuredangle CZO = 90^\circ$ and we get $\measuredangle PCQ = \measuredangle OCZ = \measuredangle OCB$, which completes the proof.

I-4 N

Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(a) + f(b)$ divides $2(a + b - 1)$ for all a, $b \in \mathbb{N}$.

Remark: $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of positive integers.

Answer. The only solutions are the constant function $f(a) = 1$ for all $a \in \mathbb{N}$ and the function $f(a) = 2a - 1$ for all $a \in \mathbb{N}$.

Solution 1. We will first prove that f is either injective or bounded.

Assume that we have $f(m) = f(n) = t$ for some m and n. If we plug in m and n as b, we get respectively:

$$f(a) + t \mid 2(a + m - 1)$$

$$f(a) + t \mid 2(a + n - 1).$$

Since the divisor of two numbers must divide their difference, we get $f(a) + t \mid 2m - 2n$. That means that if f is not bounded, it is injective, because then we must have $m = n$.

Case 1: f is injective

If we put $a = b = 1$ in the given relation, we get $f(1) = 1$. Putting $a = b$ gives us $f(a) \mid 2a - 1$. Since f is injective, we can now prove by induction that $f(a) = 2a - 1$ for $a = 2, 3, \ldots$ since all smaller divisors of $2a - 1$ are already attained by previous values of f.

Case 2: f is bounded

If f is bounded, the maximum of f exists and for any $a \in \mathbb{N}$ we can choose a prime p that is greater than a and at least three times greater than this maximum. We can now choose $b \in \mathbb{N}$ such that $a + b - 1 = p$ and we get $f(a) + f(b) \mid 2p$. Clearly, $f(a) + f(b) < p$ because of our choice of p. So we must have $f(a) + f(b) = 2$, which gives us $f(a) = 1$ for all $a \in \mathbb{N}$.

It is easily checked that the functions $f(n) = 2n - 1$ and $f(n) = 1$ satisfy the condition of the problem.

Solution 2. Setting $a = b$ gives $f(a) \mid 2a - 1$, which implies that $f(1) = 1$ and that $f(a)$ is odd for all a. Setting $a = 2$ and $b = 1$ gives $f(2) + 1 \mid 4$, therefore $f(2) = 1$ or $f(2) = 3$.

Case 1: $f(2) = 1$.

We choose $b = 1$ and $b = 2$ to obtain $f(a)+1 \mid 2a$ and $f(a)+1 \mid 2a+2$, which implies $f(a)+1 \mid 2$ and therefore $f(a) = 1$ for all a. The constant function $f(a) = 1$ is clearly a solution.

Case 2: $f(2) = 3$.

We first show that $f(a) = 2a - 1$ for $a > 1$ implies $f(a+2) = 2a + 3$. Setting $b = a + 2$ gives $f(a+2) + 2a - 1 \mid 2(2a+1)$. If $f(a+2) = 1$, then $a \mid 2a + 1$. This implies $a = 1$, which was excluded. Therefore, the odd number $f(a+2)$ is greater than 2, so $f(a+2) + 2a - 1$ is a divisor of $2(2a+1)$ that is greater than $2a + 1$ (half of $2(2a+1)$). Thus $f(a+2) + 2a - 1$ must be equal to $2(2a+1)$, which gives $f(a+2) = 2a + 3$ as desired.

Therefore, $f(2) = 3$ implies $f(4) = 7$. Now, setting $a = 3$ and $b = 4$ gives $f(3) + 7 \mid 12$, which implies $f(3) = 5$.

Since we now know that $f(a) = 2a - 1$ holds for 1, 2, 3, 4 and we can use induction in steps of two, we get $f(a) = 2a - 1$ for all a, which is clearly a solution.

The two solutions are $f(a) = 1$ for all a and $f(a) = 2a - 1$ for all a.

Solution. 3. We have $f(1) = 1$ and $f(a) \mid (2a - 1)$ as in the previous solution.

If $f(2) = 1$, then $f(a) = 1$ for all a as in the previous solution.

Therefore, we only have to consider $f(2) = 3$. We easily check that $f(a) = 2a - 1$ for all a is a solution.

Choose k maximally such that $f(a) = 2a - 1$ holds for all $1 \leqslant a \leqslant k$. Then setting $a = k$ and $b = k + 1$ yields

$$2k - 1 + f(k+1) = f(k) + f(k+1) \mid 4k,$$

which by maximality of k implies that $f(k+1) = 1$.

Setting $a = k - 1$ and $b = k + 1$ yields

$$2k - 2 = f(k-1) + f(k+1) \mid 2(k - 1 + k + 1 - 1) = 4k - 2,$$

which also implies $2k - 2 \mid ((4k-2) - 2(2k-2)) = 2$ and thus $k = 2$.

We conclude that $f(3) = 1$ and $f(4) \mid 7$. If $f(4) = 1$, then

$$4 = f(2) + f(4) \mid 10,$$

a contradiction. Thus $f(4) = 7$. This leads to the contradiction

$$8 = f(3) + f(4) \mid 12.$$

Thus there are only the constant solution and the solution $f(a) = 2a - 1$ for all a.

Solution 4. We have $f(1) = 1$ and $f(a) \mid (2a - 1)$ as in the previous solutions.

If $f(2) = 1$, then $f(a) = 1$ for all a as in the previous solutions.

Therefore, we only have to consider $f(2) = 3$.

Let p be a prime with $p \equiv -1 \pmod 4$. Since we already know that $f(a) \mid 2a - 1$, we get $f\left(\frac{p+1}{2}\right) \mid p$ which implies that $f\left(\frac{p+1}{2}\right)$ is either 1 or p.

If $f\left(\frac{p+1}{2}\right) = 1$ then we choose $a = 2$ and $b = \frac{p+1}{2}$ in the original equation and get $4 \mid p+3$ which is impossible. Therefore, $f\left(\frac{p+1}{2}\right) = p$ for all such primes p.

Now we choose $b = \frac{p+1}{2}$ in the original equation and get

$$f(a) + p \mid 2a - 1 + p \implies f(a) + p \mid 2a - 1 - f(a).$$

Since there exist arbitrarily large primes p with $p \equiv -1 \pmod 4$, the right-hand side has to be 0, so $f(a) = 2a - 1$ which is indeed a solution.

T-1 A

Determine all triples (a, b, c) of real numbers satisfying the system of equations

$$\begin{aligned} a^2 + ab + c &= 0, \\ b^2 + bc + a &= 0, \\ c^2 + ca + b &= 0. \end{aligned}$$

Answer. The solutions are

$$(a, b, c) \in \left\{ (0, 0, 0), \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right) \right\}.$$

Solution. If one of the numbers a, b and c is equal to zero, it is easy to see that the other two numbers also have to be equal to zero, which gives us the solution $(0, 0, 0)$.

Now assume that a, b, $c \neq 0$.

If all three numbers are positive, then the left-hand side of each equation is positive, while the right-hand sides are equal to zero, which is impossible.

Let us assume that only one of the numbers is positive, and without loss of generality let it be a. Since b, $c < 0$, it follows that $b^2 + bc + a > 0$, which is a contradiction.

It remains to consider the two following cases:

(a) All three numbers are negative.

We substitute $a = -x$, $b = -y$ and $c = -z$, where x, y, $z > 0$. The original system transforms into

$$x^2 + xy = z \tag{1}$$
$$y^2 + yz = x \tag{2}$$
$$z^2 + zx = y.$$

The system is cyclic, so we can assume that $x \leqslant y$ and $x \leqslant z$. Now we have

$$\begin{aligned} x^2 + xy = z \geqslant x &\implies x + y \geqslant 1, \\ y^2 + yz = x \leqslant y &\implies y + z \leqslant 1. \end{aligned}$$

From the previous two inequalities we conclude that

$$x + y \geqslant 1 \geqslant y + z, \qquad \text{i.e.} \quad x \geqslant z.$$

On the other hand $x \leqslant z$, so we get $x = z$.

Now, from equation (1) it follows that $x + y = 1$, while from equation (2) it follows that

$$x = y^2 + yz = y^2 + yx = y(y + x) = y.$$

Thus $x = y = z$ and from $x + y = 1$ we see that $x = y = z = 1/2$.

We easily verify that $(a, b, c) = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$ is indeed a solution.

(b) Exactly one of the numbers is negative.

Without loss of generality we can assume that c is negative, while a and b are positive. From the second equation we conclude that $b(b + c) = -a < 0$, thus $b + c < 0$. The third equation yields $c(a + c) = -b < 0$, thus $a + c > 0$.

Adding $a + b$ to the first equation and cyclic permutation yields

$$a + b + c = (1 - a)(a + b) = (1 - b)(b + c) = (1 - c)(c + a).$$

The last product is positive. This implies that $1 - a > 0$ and $1 - b < 0$ by our above considerations. Therefore $0 < a + c < 1 + c < b + c < 0$, a contradiction.

Solution by symmetric functions. We set

$$p := a + b + c, \quad q := ab + ac + bc, \quad r = abc. \tag{3}$$

Our strategy will be to determine p, q and r by considering equations of the form

$$\sum_{\text{cyc}} f(a, b, c)(a^2 + ab + c) = 0.$$

By setting $f(a, b, c) = 1$, we find $p^2 - q + p = 0$.

By setting $f(a, b, c) = b$, we find $pq - 3r + q = 0$.
(Here we use the general identity $\sum_{\text{cyc}}(a^2b + ab^2) = pq - 3r$.)

By setting $f(a, b, c) = c^2$, we find $(q^2 - 2pr) + pr + (p^3 - 3pq + 3r) = 0$.

By elimination of $q = p^2 + p$ and $r = \frac{pq+q}{3} = \frac{p(p+1)^2}{3}$ we find

$$(p^2 + p)^2 - p \cdot \frac{p(p + 1)^2}{3} + p^3 - 3p(p^2 + p) + p(p + 1)^2 = 0,$$

which is equivalent to

$$0 = 2p^4 + p^3 - p^2 + 3p = p(2p+3)(p^2 - p + 1).$$

Now we see that either $p = 0$ or $p = -\frac{3}{2}$.

In the case $p = 0$ we find that also $q = 0$ and $r = 0$, whence $a = b = c = 0$. In the case $p = -\frac{3}{2}$ we find $q = \frac{3}{4}$ and $r = -\frac{1}{8}$, hence a, b, c are the solutions to the cubic equation

$$0 = x^3 + \frac{3}{2}x^2 + \frac{3}{4}x + \frac{1}{8} = \left(x + \frac{1}{2}\right)^3.$$

This gives $a = b = c = -\frac{1}{2}$.

Solution. As in the first solution, we prove that as soon as one of the variables is 0, all three variables have to be 0. Obviously, the triple $(0,0,0)$ is a solution. From now on, we may therefore assume that all three variables are non-zero.

We can rewrite the system of equations as

$$\begin{aligned} -c &= a(a+b), \\ -a &= b(b+c), \\ -b &= c(c+a). \end{aligned}$$

Multiplying these equations and dividing by $abc \neq 0$ gives

$$(a+b)(b+c)(c+a) = -1. \tag{4}$$

On the other hand by summing up the equations we get

$$-a - b - c = a^2 + b^2 + c^2 + ab + bc + ac. \tag{5}$$

Now we substitute

$$x = a+b, \qquad y = b+c, \qquad z = c+a.$$

which transforms equation (4) and (5) into

$$xyz = -1, \qquad -\frac{x+y+z}{2} = \frac{x^2+y^2+z^2}{2} \tag{6}$$

Now we calculate

$$3(x^2+y^2+z^2) \geqslant (|x|+|y|+|z|)^2 \geqslant |x+y+z|^2 = (x^2+y^2+z^2)^2 \geqslant 3^2 \sqrt[3]{x^2y^2z^2}^2 = 9,$$

where the first inequality comes from Cauchy-Schwarz (or QM-AM), the second one from the triangle inequality and the last one from AM-GM. The equalities come from (6).

Now, if we denote $S = x^2 + y^2 + z^2$, we have the inequalities

$$3S \geqslant S^2 \geqslant 9,$$

and because we trivially have $S > 0$ (note that $S = 0$ would imply $x = y = z = 0$ and hence $a = b = c = 0$, which has already been excluded), we can split it into the inequalities $3 \geqslant S$ and $S \geqslant 3$, so we have equality and actually all the inequalities are equalities.

The case of equality for the triangle equality is when all nonzero x, y, z have the same sign and, in view of equations 6, the only possibility is that x, y, z are all negative. Moreover, in the last inequality, we have equality exactly when $x^2 = y^2 = z^2$ and, because they have the same sign, it means $x = y = z$. Finally, in view of $xyz = -1$, the only possibility is $x = y = z = -1$. By definition of x, y, z the values of a, b, c are then

$$(a, b, c) = \left(\frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}\right)$$

and this is indeed a solution.

T-2 A

Let $\mathbb{R}$ denote the set of real numbers. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x)f(y) = xf(f(y-x)) + xf(2x) + f(x^2)$$

holds for all real numbers x and y.

Answer. There are two solutions:

$$f\colon \mathbb{R} \to \mathbb{R}\colon x \mapsto 0, \qquad g\colon \mathbb{R} \to \mathbb{R}\colon x \mapsto 3x.$$

Solution. We set $x = 0$ and get $f(0)f(y) = f(0)$, so $f(0) = 0$ or $f(y) = 1$ for all y. The latter leads to a contradiction.

We set $y = x + z$ and get

$$f(x)f(x+z) = xf(f(z)) + xf(2x) + f(x^2) \tag{1}$$

for all x and z. Setting $z = 0$ yields

$$f(x)^2 = xf(2x) + f(x^2). \tag{2}$$

We set $C = f(1)$. For $x = 1$ and $x = 2$, we get $f(2) = C(C-1)$ and $f(4) = C^2(C-1)^2/3$, respectively.

If $C = 0$, so (1) with $x = 1$ yields $f(f(z)) = f(2) = 0$, so (1) reads $f(x)f(x+z) = xf(2x)+f(x^2)$ for all z, which implies that $f(x) = 0$ for all x.

Setting $x = 1$ and $x = 2$ in (1) leads to

$$\begin{aligned} Cf(z+1) &= f(f(z)) + C^2, \\ C(C-1)f(z+2) &= 2f(f(z)) + C^2(C-1)^2. \end{aligned}$$

Eliminating $f(f(z))$ and division by $C \neq 0$ leads to

$$(C-1)f(z+2) - 2f(z+1) = C(C^2 - 2C - 1) \tag{3}$$

for all z. Setting $z = -1$ leads to $C(C-1) = C(C^2 - 2C - 1)$. In view of $C \neq 0$, this implies $C = 3$. Inserting this in (3), dividing by 2 and shifting z leads to

$$f(z+1) - f(z) = 3$$

for all z.

We set $z = 1$ in (1) and get

$$f(x)(f(x) + 3) = 9x + xf(2x) + f(x^2).$$

Together with (2), we get $3f(x) = 9x$, i.e., $f(x) = 3x$ for all x.

Both $f(x) = 3x$ and $f(x) = 0$ are solutions.

Alternative Solution. Setting $x = 0$ in the original equation gives $f(0)f(y) = f(0)$. If $f(0) \neq 0$ then $f(x) = 1,\ x \in \mathbb{R}$, but this function does not satisfy the original equation. Hence, $f(0) = 0$.

Setting $y = 0$ and $y = x$ in the original equation we get

$$0 = xf(f(-x)) + xf(2x) + f(x^2), \qquad f(x)^2 = xf(2x) + f(x^2). \tag{4}$$

In combination this gives

$$-xf(f(-x)) = f(x)^2 \qquad \text{and} \qquad xf(f(x)) = f(-x)^2,\ x \in \mathbb{R}.$$

Multiplying the original equation by $y - x$ gives

$$(y-x)f(x)f(y) = x(y-x)f(f(y-x)) + (y-x)(xf(2x) + f(x^2)) = xf(x-y)^2 + (y-x)f(x)^2,$$

which gives

$$xf(x-y)^2 = (y-x)f(x)(f(y) - f(x)). \tag{5}$$

Now setting $x = 2y \neq 0$ in (5) we get

$$2yf(y)^2 = -yf(2y)(f(y) - f(2y))$$

and consequently

$$0 = f(2y)^2 - f(y)f(2y) - 2f(y)^2 = (f(2y) + f(y))(f(2y) - 2f(y)).$$

Thus, for any $y \neq 0$ (and for $y = 0$ as well) we have $f(2y) = -f(y)$ or $f(2y) = 2f(y)$.

Setting $y = 2x \neq 0$ in (5) we get

$$xf(-x)^2 = xf(x)(f(2x) - f(x))$$

and consequently

$$f(-x)^2 + f(x)^2 = f(x)f(2x).$$

If $f(2x) = -f(x)$ holds for some $x \in \mathbb{R}$, then $f(-x)^2 + 2f(x)^2 = 0$ implies $f(-x) = f(x) = 0 = f(2x)$. Hence, the equality $f(2x) = 2f(x)$ holds for all $x \in \mathbb{R}$.

Replacing y by $y + x$ in the original equation and using (4) gives

$$f(x)f(x+y) = xf(f(y)) + xf(2x) + f(x^2) = xf(f(y)) + f(x)^2. \tag{6}$$

Now setting $y = x$ gives $2f(x)^2 = xf(f(x)) + f(x)^2$, which means $f(x)^2 = xf(f(x))$. Multiplying (6) by y yields

$$yf(x)f(x+y) = xf(y)^2 + yf(x)^2 \tag{7}$$

Here we can deduce that $f(x) = 0$ implies $x = 0$, unless f is identically 0. Now we can interchange x and y and achieve $yf(x)f(x+y) = xf(y)f(x+y)$. Setting $y = 1$ now gives $f(x) = xf(1)$ for all $x \neq -1$. However, we also have

$$f(-1) = \frac{1}{2}f(-2) = \frac{1}{2} \cdot (-2)f(1) = -f(1),$$

so $f(x) = xf(1)$ is valid for all x. Plugging in $f(x) = cx$ easily gives $c = 0$ or $c = 3$. Hence these are the two solutions.

T-3 C

A tract of land in the shape of an 8×8 square, whose sides are oriented north–south and east–west, consists of 64 smaller 1×1 square plots. There can be at most one house on each of the individual plots. A house can only occupy a single 1×1 square plot.

A house is said to be *blocked from sunlight* if there are three houses on the plots immediately to its east, west and south.

What is the maximum number of houses that can simultaneously exist, such that none of them is blocked from sunlight?

Remark: By definition, houses on the east, west and south borders are never blocked from sunlight.

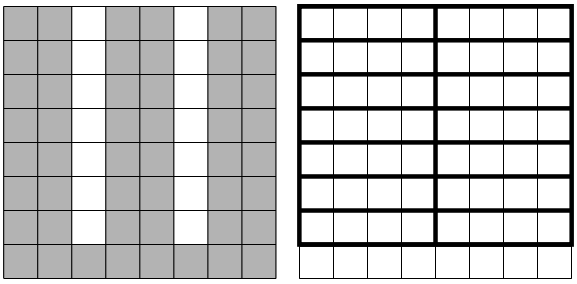
Answer. The maximal number of houses is 50.

Solution. Let us represent the tract as an 8×8-chessboard, with cells colored black if the corresponding parcel is occupied, and white otherwise. We denote by (i, j) the cell in the $i-$th row and $j-$th column (with the first row being the northernmost and the first column being the westernmost). We start by showing that an optimal configuration can be obtained by coloring all the cells along the east, south, and west borders.

Assume that there is an optimal configuration in which one of those cells, for example $(i, 1)$, is left white. Since we have an optimal configuration, this cell cannot be colored black. This means that by coloring $(i, 1)$, we would block the cell $(i, 2)$. In other words, we know that the cells $(i, 2)$, $(i, 3)$ and $(i + 1, 2)$ are all colored black in this optimal configuration. However, we now see that we can color $(i, 1)$ instead of $(i, 2)$, keeping the same number of black cells and coloring $(i, 1)$, without disturbing any of the other cells.

We can apply the same reasoning to any of the cells $(1, 1) - (8, 1)$, $(8, 1) - (8, 8)$ and, similarly, to $(8, 8) - (1, 8)$, thus showing that there is an optimal configuration in which all the cells along the E, S, W borders are colored black.

Those cells being colored, we are left with a 7×6 area of the board. We can now show that no more than 28 cells in this area can be colored black. In order to obtain 28, the average number of black cells per row has to be 4. However, if any row contains six black cells, the next row down cannot contain any black cells, since such a black cell would block the cell immediately north of it. Similarly, if a row were to contain 5 black cells, the next row down would be able to contain at most 3 black cells (namely in the cell immediately below the single white one and the two next to it). This shows that the average number of black cells per row in our 7×6 area cannot be greater than 4.



(a) (b)

Figure 1: Alternative Solution

This gives us an upper bound on the total number of black cells: the 22 border cells plus 28 cells in the remaining 7×6 area, i.e., 50 cells in total.

An example to show that 50 can indeed be achieved is the following. We color the columns $1, 2, 4, 5, 7, 8$ and row 8 black, leaving the other cells white. This coloring clearly satisfies the conditions and contains exactly 50 black cells, completing the proof.

Alternative Solution. By building a house on each dark grey plot in Figure 1(a), we see that 50 houses can be built accordingly.

We will prove that no more than 50 houses can be built, or, equivalently, that at least 14 plots remain empty. Consider the 14 4×1 rectangles in Figure 1(b) marked by their thick boundary. We will uniquely assign one empty plot to each of these 14 rectangles as follows:

- if the rectangle contains at least one empty plot, assign to it the easternmost such plot;
- if the rectangle contains no empty plots, assign to it the westernmost empty plot in the rectangle directly to the south of it.

In order to see that this assignment is feasible, note that if some rectangle contains no empty plots, then its two central houses are blocked from sunlight from the east and west. Therefore, the two central plots of the rectangle directly to the south of it must be empty, showing that we can indeed assign its westernmost empty plot to the original rectangle while leaving its easternmost empty plot unassigned. The above assignment is thus feasible and shows that there are at least 14 empty plots, concluding the proof.

T-4 C

A class of high school students wrote a test. Every question was graded as either 1 point for a correct answer or 0 points otherwise. It is known that each question was answered correctly by at least one student and the students did not all achieve the same total score.

Prove that there was a question on the test with the following property: The students who answered the question correctly got a higher average test score than those who did not.

Solution. Let n be the number of the students in the class and a their average score. Denote by P and S the set of all problems, and the set of all students resp., and let $S(p)$ be the non-empty set of students who solved problem $p \in P$. For any student s, let $\mathrm{sc}(s)$ be the score of s. For any proposition A, let $[A] = 1$ if A is true and 0 if A is false.

We will prove the assertion by contradiction. Assume that on all questions the average test score of solvers was at most the general average a, that is

$$a \geqslant \frac{1}{|S(p)|} \sum_{s \in S(p)} \mathrm{sc}(s) \Leftrightarrow a|S(p)| \geqslant \sum_{s \in S(p)} \mathrm{sc}(s).$$

We now sum these inequalities over all problems $p \in P$ to get

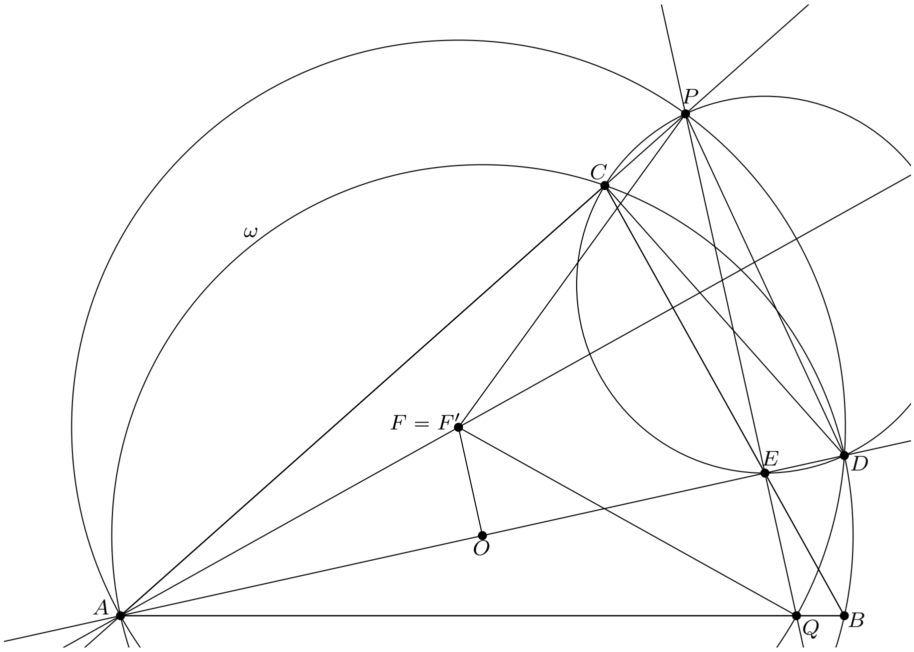
$$\begin{aligned}
& a \sum_{p \in P} |S(p)| \geqslant \sum_{p \in P} \sum_{s \in S(p)} \mathrm{sc}(s) \\
\Leftrightarrow \quad & a \sum_{p \in P} \sum_{s \in S} [s \text{ solved } p] \geqslant \sum_{p \in P} \sum_{s \in S} [s \text{ solved } p] \sum_{q \in P} [s \text{ solved } q] \\
\Leftrightarrow \quad & a \sum_{s \in S} \mathrm{sc}(s) \geqslant \sum_{s \in S} \left(\sum_{p \in P} [s \text{ solved } p] \right) \cdot \left(\sum_{q \in P} [s \text{ solved } q] \right) \\
\Leftrightarrow \quad & \frac{1}{n} \left(\sum_{s \in S} \mathrm{sc}(s) \right)^2 \geqslant \sum_{s \in S} \mathrm{sc}(s)^2.
\end{aligned}$$

However, this is the reverse of the inequality between the arithmetic and the quadratic mean. Since the case of equality, namely that $\mathrm{sc}(s)$ is the same for all $s \in S$, is excluded by the problem statement, we arrive at the desired contradiction.

T-5 G

Let ABC be an acute-angled triangle with $AB \neq AC$, and let O be its circumcentre. The line AO intersects the circumcircle ω of ABC a second time in point D, and the line BC in point E. The circumcircle of CDE intersects the line CA a second time in point P. The line PE intersects the line AB in point Q. The line through O parallel to PE intersects the altitude of the triangle ABC that passes through A in point F.

Prove that $FP = FQ$.



Solution 1. Let us denote $\measuredangle ABC$ by β and $\measuredangle BCA$ by γ. Without loss of generality, $AB > AC$, or equivalently $\beta < \gamma$, as in the figure.

Segment AD is a diameter of ω, so by Thales' theorem we have $\measuredangle DCA = 90°$. Since the quadrilateral $CEDP$ is cyclic, we get $\measuredangle PED = 90°$, which immediately gives $\measuredangle AEQ = 90°$.

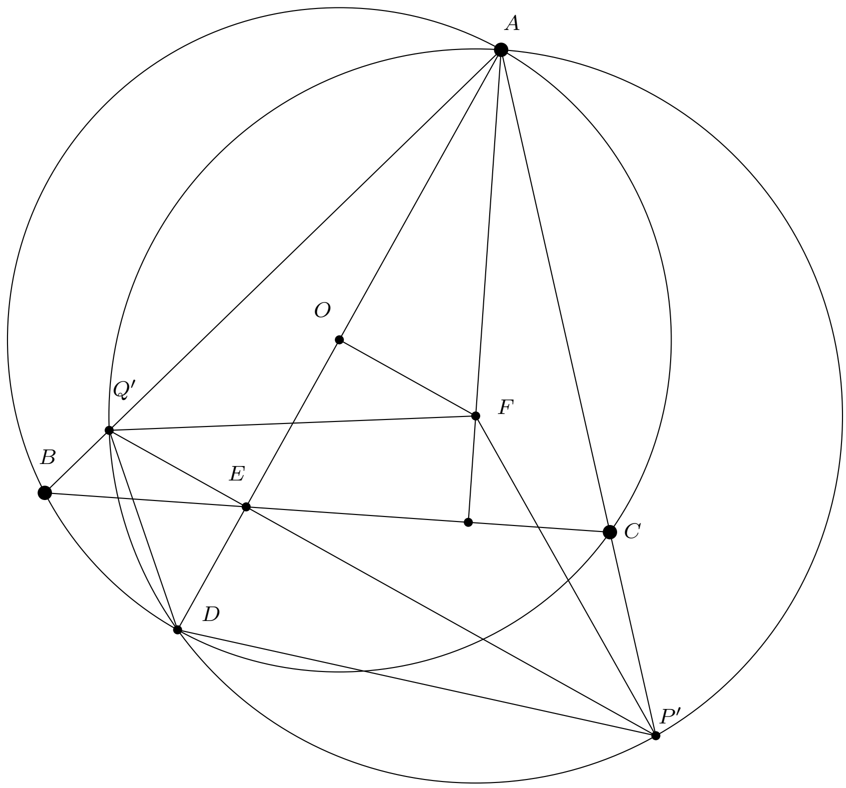
Since $\measuredangle EAQ = \measuredangle OAB = 90° - \gamma$, we also get $\measuredangle AQP = \measuredangle AQE = \gamma$. Since $CEDP$ is cyclic, we get $\measuredangle ADP = \measuredangle EDP = 180° - \measuredangle PCE = \measuredangle ACB = \gamma$. This means that the quadrilateral $AQDP$ is cyclic.

Let us denote the circumcentre of $AQDP$ by F'. We show that $F = F'$.

We have $\measuredangle APQ = 180° - \measuredangle CAB - \measuredangle AQP = \beta$. Hence $\measuredangle F'AQ = 90° - \beta$, which implies that F' lies on the altitude of ABC that passes through A.

Moreover, by definition F' must lie on the perpendicular bisector of AD, which is the line through O parallel to PE.

So we get $F' = F$, and consequently $FP = FQ$.



Solution 2. Let α, β and γ denote the angles of ABC in the natural way.

Point F is defined as the intersection point of the perpendicular bisector of diameter AD with the altitude of triangle ABC through A. We define points P' and Q' as the intersection points of the circle with center F passing through A (and therefore also through D) with sides AC and AB, respectively. First we show $P = P'$.

We calculate

$$\measuredangle EDP' = \measuredangle ADP' = \measuredangle AQ'P' = \measuredangle AQ'F + \measuredangle FQ'P' =$$

$$\measuredangle AQ'F + \frac{180^\circ - \measuredangle Q'FP'}{2} = 90^\circ - \beta + 90^\circ - \alpha = \gamma = 180^\circ - \measuredangle ECP'.$$

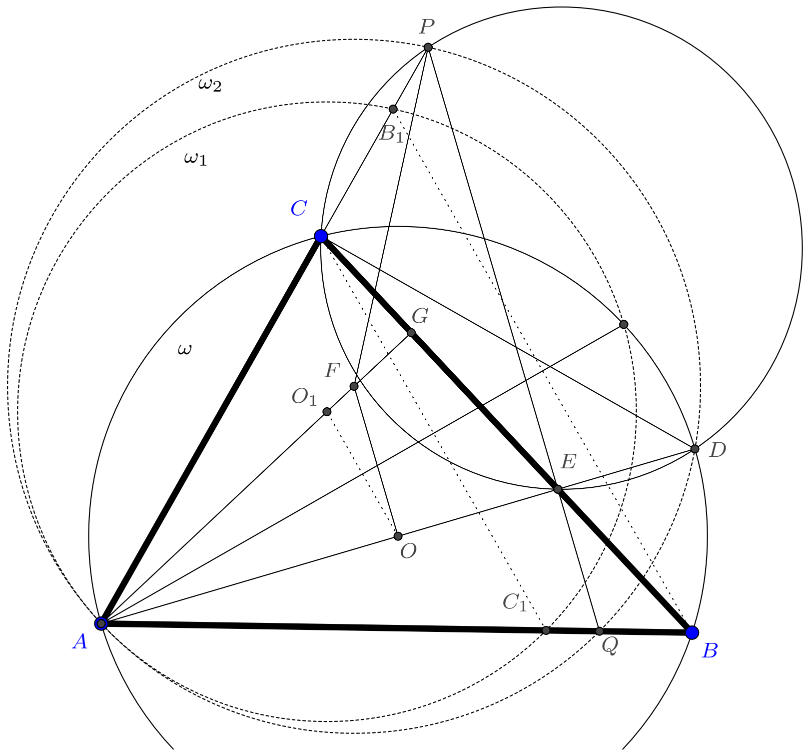
So we have that P' is the intersection of the circumcircle of triangle EDC with AC and therefore we have $P = P'$.

Now we prove $Q = Q'$. We have

$$\measuredangle ABC = \measuredangle ADC = \measuredangle EDC = \measuredangle EPC = \measuredangle Q'PA = \measuredangle Q'DA = \beta$$

and therefore quadrilateral $BDEQ'$ is cyclic. Since $\measuredangle DBQ' = 90^\circ$ we get $Q'E \perp AD$. Together with $\measuredangle DEP' = \measuredangle DEP = 90^\circ$ we have that Q' lies on the line PE and thus $Q' = Q$. Therefore we have proven $FP = FQ$.

Solution 3. Let us denote $\angle ABC$ by β and $\angle BCA$ by γ and the foot of the altitude from A by G. Without loss of generality let $AB > AC$, or equivalently $\beta < \gamma$, as in the figure.



Let ω_1 be the reflection of ω about the angle bisector of $\angle BAC$. Since $\angle EAB = \angle OAB = \angle CAG = 90^\circ - \gamma$, the center O_1 of ω_1 lies on the altitude from A and $AO = AO_1$. The circle ω_1 intersects AB for the second time at C_1 with $AC = AC_1$ and AC for the second time at B_1 with $AB = AB_1$.

Now, if we can prove

$$AC_1 : AQ = AB_1 : AP = AO_1 : AF,$$

then there is a homothety with center A which maps $C_1 \to Q$, $B_1 \to P$ and $O_1 \to F$. Hence P and Q lie on a circle ω_2 with center F and and the problem is solved.

Therefore it remains to prove $AC_1 : AQ = AB_1 : AP = AO_1 : AF$.

The triangles AOF and AGE are similar, so we have $AF = \frac{AO \cdot AE}{AG}$. Due to segment AD being a diameter of ω, we have $\angle DCA = 90^\circ$ by Thales' theorem. Since the quadrilateral $CEDP$ is cyclic, we get $\angle PED = 90^\circ$, which immediately gives $\angle AEQ = 90^\circ$. Since $\angle EAQ = \angle OAQ = 90^\circ - \gamma$, we have $AQ = \frac{AE}{\sin\gamma}$ and with $AC = \frac{AG}{\sin\gamma}$, we get

$$AC_1 : AQ = AC : AQ = \frac{AG}{\sin\gamma} : \frac{AE}{\sin\gamma} = AG : AE = AO : \frac{AO \cdot AE}{AG} = AO_1 : AF.$$

Similarly we can prove $AB_1 : AP = AO_1 : AF$ and we are ready.

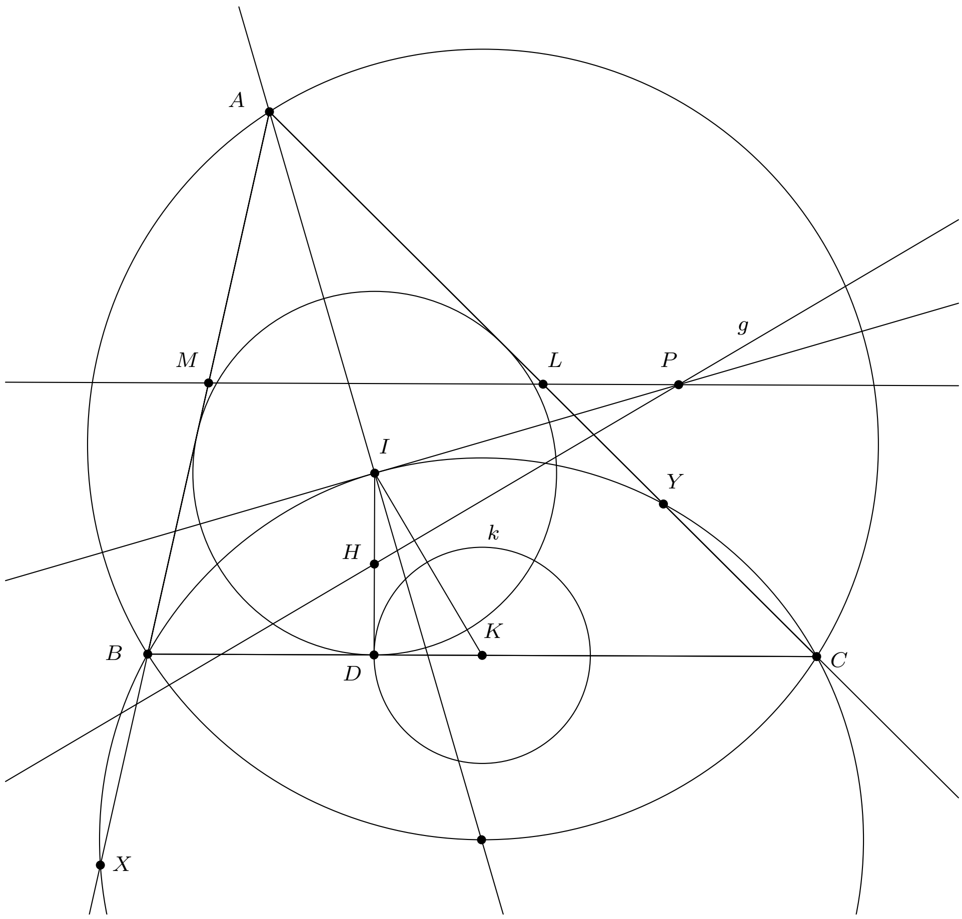
T-6 G

Let ABC be a triangle with $AB \neq AC$. The points K, L, M are the midpoints of the sides BC, CA, AB, respectively. The inscribed circle of ABC with centre I touches the side BC at point D. The line g, which passes through the midpoint of segment ID and is perpendicular to IK, intersects the line LM at point P.

Prove that $\measuredangle PIA = 90°$.

Solution 1. Let (XYZ) denote the circumcircle of a triangle XYZ. We use the following well-known lemma:

Lemma. *The centre of the circle (BIC) is the midpoint of arc BC of circle (ABC) and therefore lies on the angle bisector of $\measuredangle BAC$.*



Now assume without loss of generality that $AB < AC$, as in the figure. Let the circle (BIC) cut AB and AC at X and Y respectively. From the lemma it follows that X is symmetric to C, and Y is symmetric to B with respect to the angle bisector of $\measuredangle BAC$. We have

$$MX{\cdot}MB = (AX-AM){\cdot}MB = (AC-AM){\cdot}MB = \frac{(2AC-AB)\cdot AB}{4} = \frac{AC^2}{4} - \left(\frac{AC-AB}{2}\right)^2,$$

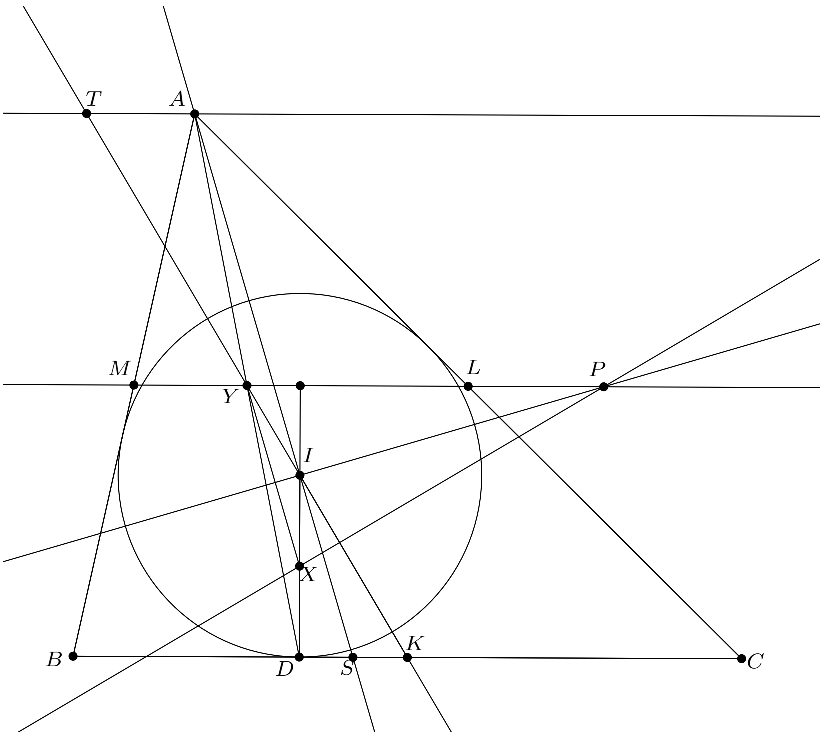
and since $\frac{AC}{2} = MK$ and

$$\frac{AC - AB}{2} = \frac{BC}{2} - \frac{AB + BC - AC}{2} = BK - BD = DK,$$

we get

$$MX \cdot MB = MK^2 - DK^2.$$

So M lies on the radical axis of the circle (BIC) and the circle k, which is the circle with midpoint K and radius DK. An analogous calculation shows that L lies on it as well, so the line through M and L is the radical axis of circles (BIC) and k. Obviously g is the radical axis of circle k and point I (regarded as a degenerate circle). Thus P is the radical centre of (BIC), k, and I. It follows that P also lies on the radical axis of (BIC) and I, which is the line perpendicular to AI passing through I. This shows that $\measuredangle PIA = 90°$.



Solution 2. Let X be the midpoint of ID, and let Y be the midpoint of AD (which is also the intersection of AD with LM). Clearly, XY is parallel to AI.

Next we show that IK passes through Y. To this end, let S be the intersection of AI with BC, and let T be the intersection of KI with the line through A that is parallel to BC. Note that triangles ATI and SKI are similar.

By the angle bisector theorem, we have

$$\frac{AT}{KS} = \frac{AI}{IS} = \frac{AB}{BS},$$

so

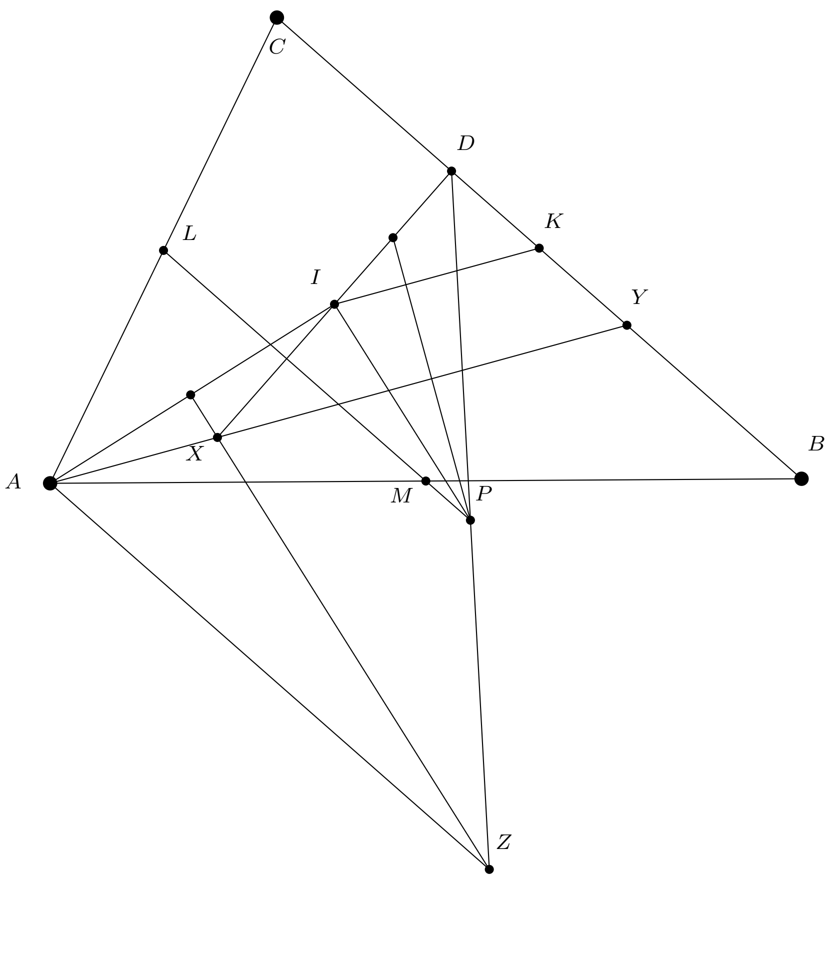
$$AT = \frac{AB \cdot KS}{BS} = \frac{AB \cdot (BK - BS)}{BS} = \frac{AB \cdot BK}{BS} - AB.$$

Using the angle bisector theorem once again, we get

$$AT = \frac{(AB + AC) \cdot BK}{BC} - AB = \frac{AB + AC}{2} - AB = \frac{BC}{2} - \frac{AB + BC - AC}{2} = BK - BD = DK.$$

Thus $DKAT$ is a parallelogram, which means that AD and KT meet at Y, the midpoint of AD.

Now we know that KI passes through Y, and by definition it is perpendicular to XP. Hence it is an altitude in XYP. Moreover, DI is clearly also an altitude in XYP (it passes through X and is perpendicular to PY). Thus I is the orthocentre of XYP, which means that PI is perpendicular to XY and thus also to AI. This proves the desired statement.



Solution 3. Let X, Y, Z be reflections of D about I, K, P respectively.

We shall prove that A, X, Y are collinear. Since K is the midpoint of segments BC, DY, we have $BD = CY$. Therefore Y is the common point of segment BC and the A-excircle. Consider homothety centered at A mapping incircle to A-excircle. It's easy to see that this homothety maps X to Y. Therefore A, X, Y are collinear.

Consider homothety with centre D and ratio 2. We easily see that $AZ \parallel LM$. Since $IX \perp BC \parallel LM$, we have

$$IX \perp AZ.$$

Moreover $IZ \parallel g \perp KI \parallel XY$. Since A, X, Y are collinear, we have

$$IZ \perp AX.$$

Thus X is the orthocentre of triangle AIZ. Therefore $ZX \perp AI$. Since $ZX \parallel PI$, we conclude that $PI \perp AI$. In other words,

$$\angle PIA = 90^\circ.$$

Solution 4. Let a, b, c be the sides of the triangle, $s = \frac{a+b+c}{2}$ the half-perimeter, ρ the radius of the inscribed circle, α, β, γ the angles and A the area of the triangle.

In the coordinate plane, let $B = (0,0)$ be the origin and $C = (a, 0)$. Then $A = (c\cos\beta, c\sin\beta)$. We get:

$$I = (s-b, \rho), \quad D = (s-b, 0), \quad H = \left(s-b, \frac{\rho}{2}\right) \text{ (midpoint of } DI\text{)}, \quad K = \left(\frac{a}{2}, 0\right).$$

The line g that passes through H and is perpendicular to IK has the following equation:

$$\left(s-b-\frac{a}{2}\right)x + \rho y = \left(s-b-\frac{a}{2}\right)(s-b) + \frac{\rho^2}{2}.$$

This is equivalent to

$$(c-b)x + 2\rho y = (c-b)(s-b) + \rho^2.$$

On the other hand, the line $h = LM$ that is parallel to the x-axis has the equation

$$y = \frac{c}{2}\sin\beta.$$

We deduce that P (the intersection of g and h) has the coordinates

$$P = \left(s-b+\frac{\rho^2 - \rho c\sin\beta}{c-b}, \frac{c}{2}\sin\beta\right).$$

Now we have to prove that

$$\overrightarrow{AI} \cdot \overrightarrow{IP} = 0,$$

which is equivalent to

$$(s-b-c\cos\beta)\cdot\frac{\rho^2-\rho c\sin\beta}{c-b}+(\rho-c\sin\beta)\Big(\frac{c}{2}\sin\beta-\rho\Big)=0.$$

Since $\rho-c\sin\beta\neq 0$, we can cancel the factor $\rho-c\sin\beta$ to obtain the equation

$$(s-b-c\cos\beta)\rho+(c-b)\Big(\frac{c}{2}\sin\beta-\rho\Big)=0,$$

which is equivalent to

$$2\rho(s-c)-2\rho c\cos\beta+c(c-b)\sin\beta=0.$$

Now we use the well-known identities $\rho=\frac{A}{s}$ and $ac\sin\beta=2A$ to end up with the equation

$$a(s-c)-ac\cos\beta+s(c-b)=0.$$

Since this is equivalent to

$$b^2=a^2+c^2-2ac\cos\beta,$$

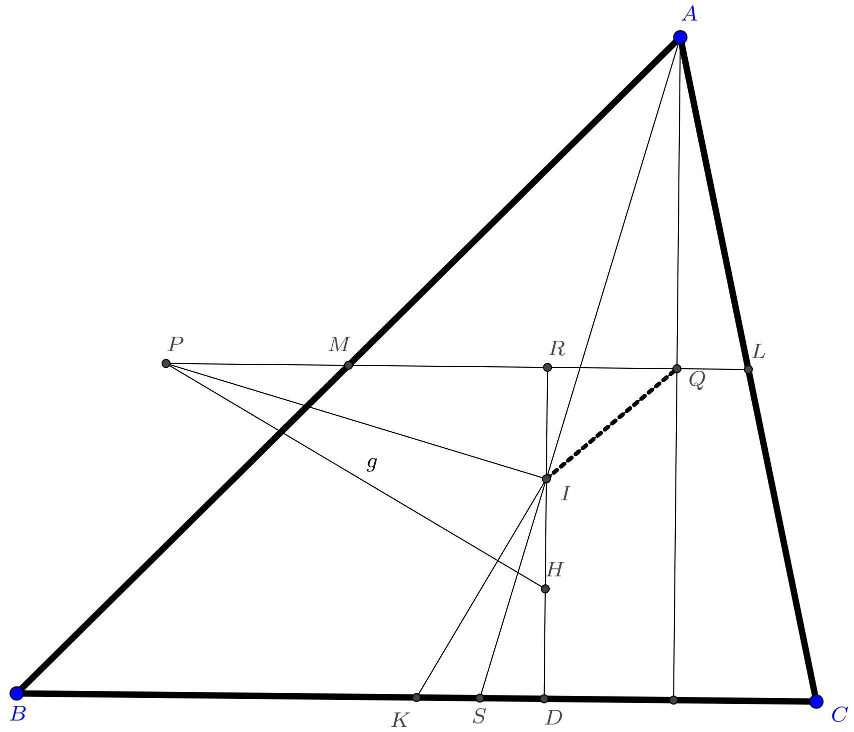
which holds by the law of cosines, the proof is complete.

Solution 5. Let S be the point of intersection of the interior angle bisector of $\angle BAC$ and BC, H the midpoint of ID, R the point of intersection of LM and ID, Q the point of intersection of LM and the altitude from A, r the inradius and h the length of the altitude from A.

Since KD and PR are perpendicular to RD and IK is perpendicular to PH, we have $\angle IKS=\angle IHP$ and $\angle RPH=\angle DIK$. So the triangles RPH and DIK are similar. We have $PH:HR=IK:KD$ and since $HR=\frac{h-r}{2}$ and $KD=\frac{a}{2}-\frac{a+b-c}{2}=\frac{c-b}{2}$ we conclude

$$PH=\frac{IK(h-r)}{(c-b)}.$$

Now we want to show that the triangles IKS and PHI are similar. Since $\angle IKS=\angle PHI$, it suffices to prove that $IK:KS=PH:HI$.



With the angle bisector theorem we get $SC = \frac{ab}{b+c}$ and we deduce
$KS = \frac{a}{2} - \frac{ab}{b+c} = \frac{a(c-b)}{2(b+c)}$.

Now we have

$$PH : HI = IK : KS \iff \frac{IK(h-r)}{(c-b)} : \frac{r}{2} = IK : \frac{a(c-b)}{2(b+c)} \iff r(a+b+c) = ah$$

and we are done, because $r(a+b+c) = ah =$ twice the area of the triangle ABC.

Hence the triangles IKS and PHI are similar and we conclude that the triangles PIR und ISD are similar too. Now we have

$$\angle QPI = \angle RPI = \angle DIS = \angle QAI$$

and it follows that $APIQ$ is cyclic, and consequently

$$90^\circ = \angle PQA = \angle PIA.$$

T-7 N

A positive integer n is called a *Mozartian number* if the numbers 1, 2, $\ldots$, n together contain an even number of each digit (in base 10).

Prove:

(a) All Mozartian numbers are even.

(b) There are infinitely many Mozartian numbers.

Solution 1.

(a) Note that we need an even number of digits alltogether if every digit occurs an even number of times. There is an odd number of numbers with one digit. For $k > 1$, there are $9 \cdot 10^{k-1}$ numbers with k digits, which is an even number. Thus we need to end after a segment of odd length of numbers with an odd number of digits, i.e., we end on an even number, so a Mozartian number is indeed even.

(b) The numbers $n = \underbrace{2\ldots2}_{2\ell}0$ are Mozartian numbers for all natural numbers ℓ: There are an even number of least significant digits 0, 1, $\ldots$, 9; and all other digits at higher positions except for those in n are repeated 10 times in a row which does not change the parities of occurrences. The leading 2ℓ digits 2 of n do not change parities, either.

Solution 2.

(a) Let k be any integer $\geqslant 0$. In the pairing $(2,3)$, $(4,5)$, ..., $(2k, 2k+1)$, the members of each pair need the same number of digits, so each pair needs an even number of digits together, so alltogether the numbers from 1 to $2k+1$ need an odd number of digits. Therefore, any Mozartian number has to be even because the total number of digits used up to a Mozartian number has to be even.

(b) We will show that $10^{2k} + 22$ are Mozartian numbers for all natural numbers k.

We first note that by the proof of the first part, we know that we need an odd number of digits up to $10^{2k} + 21$, and therefore an even number of digits up to $10^{2k} + 22$. So it is sufficient to check that the digits $1, 2, \ldots, 9$ occur an even number of times because the condition for 0 will be automatically satisfied.

Now, we will consider the numbers from 0 to $10^{2k} - 1$ as numbers with $2k + 1$ digits with leading zeros where necessary. Clearly, each digit must occur equally often. Since the number of all digits in this list is divisible by 100, this quantity is still divisible by 10, therefore even. This proves that nonzero digits occur an even number of times in this interval.

It remains to consider the numbers 10^{2k}, $10^{2k}+1$, ..., $10^{2k}+22$. Clearly, the leading ones occur an odd number of times. Since the list $1, 2, \ldots, 22$ contains an odd number of ones and an even number of the other digits, the proof is finished.

Solution 3. (only Part (b))

We will first show that for any $k \geqslant 1$ the numbers from 0 to $20k-1$ together contain an even number of each digit from 0 to 9.

The units digits clearly run from 0 to 9 an even number of times, so they contribute an even number to each digit count. For any possible fixed choice of all digits except the units digits, there are 10 numbers that satisfy this condition, so again, they contribute an even number to each digit count which proves the assertion.

Consider now the numbers from 1 to $M = 20k$ where M has a decimal representation that contains an odd number of zeros and an even number of each digit from 1 to 9. Since the odd number of zeros compensates for the missing zero that was counted in the above assertion, we find that M is a Mozartian number.

There are clearly infinitely many such numbers, for example all numbers of the form $22\ldots20$ that contain an even number of 2s.

Comment. The argument of Solution 3 shows that Mozartian numbers that are multiples of 20 are exactly those multiples that contain an odd number of 0s and an even number of all other digits.

By an analogous argument, one can now find all Mozartian numbers. The following table lists the parity restrictions on the digit counts for each possible even residue modulo 20 where e stands for even and o stands for odd. The rows list the different possible residues and the columns lists the digits from 0 to 9. For example, 10198 is a Mozartian number because it has residue 18 modulo 20 and the digits that occur an odd number of times are 0, 8 and 9. These conditions are the only restrictions on Mozartian numbers.

	0	1	2	3	4	5	6	7	8	9
0	o	e	e	e	e	e	e	e	e	e
2	e	o	e	e	e	e	e	e	e	e
4	e	o	o	o	e	e	e	e	e	e
6	e	o	o	o	o	o	e	e	e	e
8	e	o	o	o	o	o	o	o	e	e
10	e	o	o	o	o	o	o	o	o	o
12	o	e	o	o	o	o	o	o	o	o
14	o	e	e	e	o	o	o	o	o	o
16	o	e	e	e	e	e	o	o	o	o
18	o	e	e	e	e	e	e	e	o	o

T-8 N

We consider the equation $a^2 + b^2 + c^2 + n = abc$, where a, b, c are positive integers.

Prove:

(a) There are no solutions (a, b, c) for $n = 2017$.

(b) For $n = 2016$, a must be divisible by 3 for every solution (a, b, c).

(c) The equation has infinitely many solutions (a, b, c) for $n = 2016$.

Solution 1.

(a) We distinguish cases depending on the parity of a, b, c:

- If all three are odd, we have $a^2 + b^2 + c^2 + 2017 \equiv 0 \pmod 2$ and $abc \equiv 1 \pmod 2$.
- If exactly one of them is even, we have $a^2 + b^2 + c^2 + 2017 \equiv 1 \pmod 2$ and $abc \equiv 0 \pmod 2$.
- If exactly two of them are even, we have $a^2 + b^2 + c^2 + 2017 \equiv 2 \pmod 4$ (recalling that squares are either congruent to 0 or 1 modulo 4) and $abc \equiv 0 \pmod 4$.
- If all three are even, we have $a^2 + b^2 + c^2 + 2017 \equiv 1 \pmod 2$ and $abc \equiv 0 \pmod 2$.

In each of the four cases, we see that the two sides of the equation cannot be equal.

(b) Note that $m^2 \equiv 0 \pmod 3$ if m is divisible by 3 and $m^2 \equiv 1 \pmod 3$ otherwise, and note also that 2016 is divisible by 3. We consider two cases:

- If none of the three numbers a, b, c is divisible by 3, then neither is abc, while on the other hand $a^2 + b^2 + c^2 + 2016 \equiv 1 + 1 + 1 + 0 \equiv 0 \pmod 3$. Hence we get a contradiction.
- Otherwise, abc is divisible by 3, so $a^2 + b^2 + c^2$ must be divisible by 3 as well. If exactly one of the three variables is divisible by 3, we have $a^2 + b^2 + c^2 \equiv 2 \pmod 3$, and if exactly two of them are divisible by 3, we have $a^2 + b^2 + c^2 \equiv 1 \pmod 3$. In both cases, we see that there cannot be a solution. This leaves us with the only possibility that a, b, c are all divisible by 3.

(c) We know from the previous part that we must have $a = 3x$, $b = 3y$, $c = 3z$ for certain positive integers x, y, z. We plug these into the given equation and divide by 9 to obtain

$$x^2 + y^2 + z^2 + 224 = 3xyz.$$

Note that $225 = 15^2$ is a perfect square, so we try to find solutions with $x = 1$:

$$y^2 + z^2 + 225 = 3yz.$$

Indeed, $y = z = 15$ is a solution, and we find further solutions by means of "Vieta jumping". Suppose that (y_0, z_0) is a solution, i.e.,

$$y_0^2 + z_0^2 + 225 = 3y_0z_0,$$

where $y_0 \geqslant z_0$. The second solution to the quadratic equation

$$z^2 - 3y_0z + (225 + y_0^2) = 0$$

is $z_1 = 3y_0 - z_0 \geqslant 2y_0 > y_0$, giving us a new solution pair (z_1, y_0) that has a greater first component than the previous one. Repeating the procedure, we obtain infinitely many solutions.

Solution 2. The third part can also be solved by means of the theory of Pellian equations. Let us return to the equation

$$y^2 + z^2 + 225 = 3yz.$$

We multiply by 4 and complete the square:

$$4y^2 - 12yz + 4z^2 + 900 = (2y - 3z)^2 - 5z^2 + 900 = 0.$$

For odd k, we have

$$(2 + \sqrt{5})^k \cdot (2 - \sqrt{5})^k = -1.$$

We can write $(2 + \sqrt{5})^k$ as $u + v\sqrt{5}$ for certain positive integers u and v, so that $(2 - \sqrt{5})^k = u - v\sqrt{5}$ and thus

$$u^2 - 5v^2 = -1.$$

Now simply set $z = 30v$ and $y = 15u + 45v$ (so that $2y - 3z = 30u$) to obtain

$$(2y - 3z)^2 - 5z^2 + 900 = 0,$$

as desired. Since we obtain a solution for every odd k in this way, there must be infinitely many.