11TH MIDDLE EUROPEAN MATHEMATICAL OLYMPIAD
VILNIUS 2017 LITHUANIA

Contest problems with solutions

Jury & Problem Selection Committee

VILNIUS UNIVERSITY
FACULTY OF MATHEMATICS AND INFORMATICS

Countries	**Selected problems**			
	Algebra	**Combinatorics**	**Geometry**	**Number theory**
Austria	T-2	T-3		T-8
Croatia		T-4	T-5	
Czech Republic		I-2		
Germany				I-4
Poland	T-1			T-7
Slovakia	I-1		I-3, T-6	

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Individual Competition

I - 1

Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^2 + f(x)f(y)) = xf(x+y)$$

for all real numbers x and y.

Answer: $f(x) \equiv 0$, or $f(x) \equiv x$, or $f(x) \equiv -x$.

Solution. Put $x := 0$. Then $f(f(0)f(y)) = 0$, so there is at least one real number a such that $f(a) = 0$.

Let z be an arbitrary real number, let's put $x := a$, $y := z - a$. Then we get $f(a^2) = a \cdot f(z)$. If $a \neq 0$, we'll get that f is constant, i. e. $f(x) \equiv c$. The equation (1) is then equivalent to $c = cx$ for all x, which gives us $c = 0$. The function $f(x) \equiv 0$ is really a solution.

So we've proved that $f(x) = 0$ if and only if $x = 0$. Therefore after putting $y := -x$ we get

$$x^2 + f(x)f(-x) = 0 \tag{1}$$

Now let's put $y := 0$, which gives us

$$f(x^2) = xf(x) \tag{2}$$

From (3) for $x := -x$ we get $f(x^2) = -xf(-x)$, so $x(f(x) + f(-x)) = 0$, which for $x \neq 0$ means $f(x) = -f(-x)$ and for $x = 0$ it holds too, so f is an odd function.

By putting $f(-x) = -f(x)$ to (2) we get $x^2 = f(x)^2$, which means that for every x is $f(x) = x$ or $f(x) = -x$. We'll prove that $f(x) \equiv x$ or $f(x) \equiv -x$.

Assume there are a, b such that $f(a) = a$ and $f(b) = -b$. Let's put $x := a$, $y := b$ to (1). Then we have

$$f(a^2 - ab) = a \cdot f(a+b)$$

This relation together with $f(x) = x$ or $f(x) = -x$ means that

$$e_1(a^2 - ab) = e_2(a^2 + ab)$$

where $e_1, e_2 \in \{1, -1\}$. After checking options (if we WLOG suppose $e_1 = 1$, there'll be only two) we'll find out it must hold $a = 0$ or $b = 0$. If $a = 0$, then for all $x \neq 0$ we have $f(x) = -x$, which is true even for $x = 0$, so $f(x) \equiv -x$, in that case. Analogously, if $b = 0$, then $f(x) \equiv x$. These two solutions really satisfy (1).

The equation has 3 solutions: $f(x) \equiv 0$, $f(x) \equiv x$, $f(x) \equiv -x$. ◀

Alternative solution 1. Suppose, f is injective; that is, $f(y_0) = f(y_1)$ implies $y_0 = y_1$. Plugging $x = 1$ gives

$$f(1 + f(1)f(y)) = f(1 + y).$$

From injectivity, this gives $1 + f(1)f(y) = 1 + y$. We see that $f(1) = c \neq 0$, and $f(y) = c^{-1}y$. For $y = 1$ this gives $c^2 = 1$. Thus, $f(y) = y$ or $f(y) = -y$, and both are the solutions.

Assume, f is not injective, and $f(y_0) = f(y_1)$ for a certain pair $y_0 \neq y_1$. Plugging $y \mapsto y_0$, and $y \mapsto y_1$, and comparing, gives

$$xf(x + y_0) = xf(x + y_1).$$

Thus, if $x \neq 0$, then $f(x+y_0) = f(x+y_1)$. This is also valid for $x = 0$. So, for $P = y_0 - y_1 \neq 0$, we get

$$f(x + P) = f(x).$$

Put in the initial equation $x = 0$. We see that $f(f(y)f(0)) = 0$. Let a be such that $f(a) = 0$, which does exist. Put $x = a$, $y = z - a$. This gives $f(a^2) = af(z)$. If $a \neq 0$, we get that $f(x) = c$, and only $c = 0$ does satisfy this. So, if f is not identically 0, a can be only 0. But this now contradicts to $f(P) = f(0) = 0$. ◀

Alternative solution 2. Obviously, the functions $f(x) = 0$ and $f(x) = \pm x$ are solutions. Let us prove that there are no more solutions.

Let f be any other solution. Denote $f(0) = c$. Taking $x = 0$ we get $f(cf(y)) = 0$ for all y. Hence $f(a) = 0$ for $a = cf(1)$ and therefore $0 = f(cf(a)) = f(0) = c$.

Now take $y = 0$; then $f(x^2) = xf(x)$ for all x and therefore $xf(x) = -xf(-x)$, i.e. $f(-x) = -f(x)$ for all $x \neq 0$. Hence f is an odd function.

Next prove that 0 is the only point, where $f(x) = 0$. Let $f(c) = 0$ for $c \neq 0$. Then $f(c^2) = cf(y)$ for all y, i.e. $f(y) = c_1$ for all y. Obviously, such f is not a solution; we got a contradiction.

Now take $y = -x$; then $f(x^2 - f^2(x)) = 0$ and therefore $x^2 = f^2(x)$ for all x.

Suppose $f(1)f(y) = -y$ for some $y \neq 0$; then

$$f(1 - y) = f(1 + y),$$

which yields $1 - y = \pm(1 + y)$, a contradiction. ◀

I - 2

Let $n \geqslant 3$ be an integer. A labelling of the n vertices, the n sides and the interior of a regular n-gon by $2n+1$ distinct integers is called *memorable* if the following conditions hold:

1. Each side has a label that is the arithmetic mean of the labels of its endpoints.
2. The interior of the n-gon has a label that is the arithmetic mean of the labels of all the vertices.

Determine all integers $n \geqslant 3$ for which there exists a memorable labelling of a regular n-gon consisting of $2n+1$ consecutive integers.

Solution. We prove that the desired n's are precisely those divisible by 4.

Fix n and assume such labelling exists. Without loss of generality, the labels form a set $\{0, 1, \ldots, 2n\}$. A maximum can't be obtained by averaging, so number $2n$ labels a vertex. In order for the side labels to be integers, the vertex labels have to have the same parity, hence all of them are even.

Since the label of the interior is the average of the vertex as well as edge labels, we see that the interior label is the average of all labels. Thus the interior label is equal to n.

There are $n+1$ even labels and n of them label vertices. Denote by e the one that doesn't. We have

$$n = \frac{0 + 2 + \cdots + 2n - e}{n} = \frac{n(n+1) - e}{n}$$

which gives $n = e$, i.e. n has to be even. In that case, the vertex labels form a set $V = \{0, 2, \ldots, n-2, n+2, \ldots, 2n\}$ and hence the side labels form a set $S = \{1, 3, \ldots, 2n-1\}$.

Now assume $n = 4k+2$ for some integer $k \geqslant 1$. Then V contains $n/2+1$ numbers divisible by four, hence two such labels are used on neighbouring vertices which contradicts the fact that all edges get odd label. Therefore, n is divisible by 4.

Finally, for any $n = 4k$ we construct a satisfying labelling: Label the vertices by numbers

$$0, 2, 4, \ldots, 4k-2, \quad 4k+4, 4k+2, \quad 4k+8, 4k+6, \quad \ldots, \quad 8k, 8k-2$$

in this order. Then all the even labels but $n = 4k$ are used for vertices, n itself is used for the interior, and the side labels are $1, 3, \ldots, 4k-3, 4k+1, 4k+3, \ldots, 8k-1, 4k-1$ in this order.

Remark. Another construction of a satisfying labelling: Label the vertices by numbers

$$0, 2, \ldots, 2k-2, \quad 6k, 6k-2, \ldots, 4k+2, \quad 8k, 8k-2, \ldots, 6k+2, \quad 2k, 2k+2, \ldots, 4k-2$$

in this order. ◀

I - 3

Let $ABCDE$ be a convex pentagon. Let P be the intersection of the lines CE and BD. Assume that $\angle PAD = \angle ACB$ and $\angle CAP = \angle EDA$. Prove that the circumcentres of the triangles ABC and ADE are collinear with P.

Solution. Simple angle chasing gives us:

$$\angle BCD + \angle EDC = \angle ACB + \angle ACD + \angle EDA + \angle ADC =$$

$$= \angle PAD + \angle ACD + \angle CAP + \angle ADC = 180°,$$

so $BC \parallel DE$. Therefore there exists a homothety H centered in P that maps BC to DE. Let A' be the image of A under this homothety. Then simply $\angle A'ED = \angle ACB = \angle A'AD$, so quadrilateral $A'DEA$ is cyclic. This means that the circumcircle of triangle AED is the same as the circumcircle of triangle $DA'E$. But triangle ABC maps to the triangle $A'DE$, so their circumcenters are collinear with the center of homethety P, which concludes the proof. ◀

I - 4

Determine the smallest possible value of

$$|2^m - 181^n|,$$

where m and n are positive integers.

Answer: $|2^{15} - 181^2| = 7$.

Solution. Calculating

$$181^2 = 32,761$$

one should get the idea that this may be close to

$$2^{15} = 32,768$$

so taking the difference of both we arrive at the minimum possible value 7.

As we can clearly see that the difference must be positive and odd, we only need to eliminate the possibilities $1, 3$ and 5 for the given difference.

First consider the difference modulo 15 since this will lead to a short period of basis 2 and an even shorter one of basis 181. Since $2^m \equiv 1, 2, 4, 8$ modulo 15 and $181^n \equiv 1$ modulo 15 we get that

$$2^m - 181^n \equiv 0, 1, 3, 7 \, \text{modulo } 15$$

thus leaving us with possible residues $0, 1, 3, 7, 8, 12, 14$ for the absolute of the difference. Therefore we can eliminate the minimum 5 for the difference, leaving us with 1 and 3 as possible values less than 7.

We can easily see that indeed $m \geqslant 4$ is clearly required for the difference to be anywhere near 7 or less. So now let us consider the following equations:

$$2^m - 181^n = -1 \Rightarrow 2^m \equiv 181^n - 1 \equiv 0 \, \text{modulo } 3, \text{ which is impossible;}$$
$$2^m - 181^n = 1 \Rightarrow 2^m \equiv 181^n + 1 \equiv 2 \, \text{modulo } 4, \text{ which is impossible for } m \geqslant 2;$$
$$2^m - 181^n = -3 \Rightarrow 2^m \equiv 181^n - 3 \equiv 2 \, \text{modulo } 4, \text{ which is still impossible.}$$

Therefore one last equation needs to be considered:

$$2^m - 181^n = 3 \Leftrightarrow 2^m = 181^n + 3.$$

By looking at the period of the values obtained modulo 15 we can see that $m \equiv 2$ modulo 4 is required, so $m = 4k + 2$.

But then we can look at the equation modulo 13 and see that:

$$\begin{aligned}2^m - 181^n = 3 &\Leftrightarrow 2^m = 181^n + 3\\ &\Rightarrow 2^{4k+2} \equiv (-1)^n + 3 \quad \text{modulo } 13\\ &\Leftrightarrow 4 \cdot 16^k \equiv (-1)^n + 3 \quad \text{modulo } 13\\ &\Leftrightarrow 4 \cdot 3^k \equiv (-1)^n + 3 \quad \text{modulo } 13\end{aligned}$$

We can clearly see that $(-1)^n + 3 \equiv 2, 4$ modulo 13 and $4 \cdot 3^k \equiv 12, 10, 4$ modulo 13 periodically. So the only possible solution would be if $(-1)^n + 3 \equiv 4$ but that requires n to be even, thus $n = 2q$ and hence

$$|2^m - 181^n| = |2^{4k+2} - 181^{2q}| = |(2^{2k+1} - 181^q) \cdot (2^{2k+1} + 181^q)| \geqslant 183$$

so this will not lead to any solution less than 7, which proves 7 to be the minimum possible value. ◀

Team Competition

T - 1

Determine all pairs of polynomials (P, Q) with real coefficients satisfying

$$P(x + Q(y)) = Q(x + P(y))$$

for all real numbers x and y.

Answer: Either $P \equiv Q$ or $P(x) = x + a$ and $Q(x) = x + b$ for some real numbers a, b.

Solution. If either P or Q is constant then clearly $P \equiv Q$. Suppose neither of P, Q is constant.

Write $P(x) = ax^n + bx^{n-1} + R(x)$ and $Q(x) = cx^m + dx^{m-1} + S(x)$ with $n, m \geqslant 1$, $a \neq 0 \neq c$, $\deg R < n - 1$, $\deg S < m - 1$.

Then the degree of $P(x + Q(y))$ (with respect to x) is n and the leading coefficient is equal to a. Similarly, the degree of $Q(x + P(y))$ is m and the leading coefficient is c. It follows that $m = n$ and $a = c$.

Now, the x^{n-1}-coefficient of the polynomial $P(x+Q(y))$ is $anQ(y)+b$ and the corresponding coefficient of $Q(x + P(y))$ is equal to $anP(y) + d$. It follows that $anQ(y) + b = anP(y) + d$ for every $y \in \mathbb{R}$.

Putting $t = \frac{b-d}{an}$ we have $P(x) = Q(x)+t$ for every x. If $t = 0$ then $P \equiv Q$. Such polynomials satisfy required conditions.

Suppose that $t \neq 0$. Substituting $P(x) = Q(x) + t$ to $P(x + Q(y)) = Q(x + P(y))$ we get $Q(x + Q(y)) + t = Q(x + Q(y) + t)$. Putting $x := x - Q(y)$ we have $Q(x) + t = Q(x + t)$. This implies that $Q(kt) = Q(0) + kt$ for $k \in \mathbb{Z}$ and therefore $Q(x) = Q(0) + x$ for every x. It follows that $P(x) = x + Q(0) + t$. Those polynomials clearly satisfy required conditions. ◀

Alternative solution. Substitute $x = -P(y)$. We get $P(Q(y)-P(y)) = Q(0)$. If $Q(y)-P(y)$ is not a constant, then $Q(y) - P(y)$ is a polynomial which takes infinitely many values, which would imply that$P(x) = Q(0)$ with infinitely many x, hence $P(x)$ is a constant. In this case $P \equiv Q \equiv d$, where d is a constant, which is a valid solution.

Therefore $Q(y) - P(y) = c$, where c is a constant. If $c = 0$, then we get $P \equiv Q$, which is also a valid solution.

If $c \neq 0$, we get $P(x + P(y) + c) = P(x + P(y)) + c$. Let's call $z = x + P(y)$, z is any real number and $P(z + c) = P(z) + c$.

Substituting $z = 0$ and $z = c$ gives $P(c) = P(0) + c$, $P(2c) = P(c) + c = P(0) + 2c$ and it is easy to see by induction that $P(kc) = P(0) + kc$ for every positive integer k.

Let's call $G(x) = P(x) - x - P(0)$, then G is a polynomial which has infinitely many roots and so $G \equiv 0$ and $P(x) = x + a$ for all real x and a constant a. Then $Q(x) = P(x) + c = x + b$ with a constant b. Those polynomials clearly satisfy required conditions. ◀

T - 2

Determine the smallest possible real constant C such that the inequality

$$|x^3 + y^3 + z^3 + 1| \leqslant C|x^5 + y^5 + z^5 + 1|$$

holds for all real numbers x, y, z satisfying $x + y + z = -1$.

Answer: The smallest constant C is $\dfrac{9}{10}$.

Solution. The key for our solution is the replacement of 1 by $-(x+y+z)^3$ and $-(x+y+z)^5$ on the LHS and RHS, resp., of the inequality under consideration. Thus we have to deal with the equivalent inequality

$$|x^3 + y^3 + z^3 - (x + y + z)^3| \leqslant C \cdot |x^5 + y^5 + z^5 - (x + y + z)^5|.$$

Now, for instance $z = -x$ shows that the expressions on either side of our inequality are 0. Therefore, both expressions have $(x + z)$ as a factor. Similarly, $(x + y)$ and $(y + z)$ are factors, too. A bit of moderate algebra yields

$$|3(x + y)(x + z)(y + z)| \leqslant C \cdot |5(x + y)(x + z)(y + z)(x^2 + y^2 + z^2 + xy + xz + yz)|.$$

Therefore, $(x + y)(x + z)(y + z) = 0$ certainly implies equality in our inequality. We thus let further on be $(x + y)(x + z)(y + z) \neq 0$ and thus have to deal with

$$3 \leqslant 5C \cdot |x^2 + y^2 + z^2 + xy + xz + yz|.$$

Because of $|x^2 + y^2 + z^2 + xy + xz + yz| = x^2 + y^2 + z^2 + xy + xz + yz$, this is equivalent to

$$\begin{aligned} & \frac{3}{5C} \leqslant x^2 + y^2 + z^2 + xy + xz + yz, \\ \Leftrightarrow \qquad & \frac{6}{5C} \leqslant x^2 + y^2 + z^2 + (x + y + z)^2, \\ \Leftrightarrow \qquad & \frac{6}{5C} - 1 \leqslant x^2 + y^2 + z^2. \end{aligned}$$

Now, the arithmetic-square root-inequality implies

$$x^2 + y^2 + z^2 \geqslant \frac{(x + y + z)^2}{3} = \frac{1}{3}$$

with equality iff $x = y = z = -\frac{1}{3}$. Thus, a constant that works for all values of x, y, z satisfies

$$\frac{6}{5C} - 1 \leqslant \frac{1}{3},$$

that is finally
$$C \geqslant \frac{9}{10}.$$
As a summary, we have $C_{min} = \frac{9}{10}$ and there occurs equality in our inequality iff
$$(x+y)(x+z)(y+z) = 0 \quad \text{or} \quad x = y = z = -\frac{1}{3}. \qquad ◀$$

Alternative solution. We replace $z = -1 - x - y$ and factor both sides to obtain
$$|3(x+1)(y+1)(x+y)| \leqslant 5C|(x+1)(y+1)(x+y)|(x^2+xy+y^2+1+x+y).$$
If $x = -1$, $y = -1$ or $x = -y$, both sides are zero; otherwise we may cancel the corresponding factors. We have
$$(x^2+xy+y^2+1+x+y) = \left(x+\frac{y}{2}+\frac{1}{2}\right)^2 + \frac{3}{4}\left(y+\frac{1}{3}\right)^2 + \frac{2}{3},$$
so the right hand side is at least $\frac{2}{3}$. Thus $10 \cdot \frac{C}{3} \leqslant 3$, i.e., $C = \frac{9}{10}$ is the optimal value. ◀

T - 3

There is a lamp on each cell of a 2017×2017 square board. Each lamp is either on or off. A lamp is called *bad* if it has an even number of neighbours that are on. What is the smallest possible number of bad lamps on such a board?

(Two lamps are neighbours if their respective cells share a side.)

Answer: The smallest possible number of lamps with an even number of neighbours that are on is 1.

Solution. Please consult the figures at the end of this solution.

We divide the square in 1×1-squares and color the square in checkerboard fashion such that the corners are black and we call lamps on black and white squares black and white lamps, respectively. We assign the number 1 to a lamp that is on, and the number 0 to a lamp that is off.

If we assign coordinates $(0, 0)$ to the lamp in the center, we see that the black lamps are exactly the lamps with the coordinates (i, j) where $i + j$ is even.

Now we assume that the minimum number is 0 that is, there is a configuration where every lamp has an odd number of neighbours that are on, and we try to get a contradiction. For every black lamp with coordinates (i, j), i and j even, we add the numbers associated to its neighbours, and add all these numbers. The parity of this sum S can be determined in the following two ways:

On the one hand, we know that every lamp has an odd number of neighbours with value 1, so we simply have to determine the number modulo 2 of lamps with i and j even. Since we can group lamps at (i, j) with lamps at $(-i, -j)$ and the lamp in the center is the only one left, we get that S is odd.

On the other hand, every white lamp enters the sum as often as it has neighbours with i and j even. But there are exactly two such lamps because exactly one of the coordinate of the white lamp is odd and can be modified with plus or minus 1 to get a neighbour with two even coordinates. There are no problems at the boundary because this process will not change the coordinate ± 1008 so we will stay inside the square. Therefore, S is even, which is clearly a contradiction.

So, it is impossible that all lamps have an odd number of neighbours that are on.

Now, we will provide a concrete arrangement where all lamps except for the lamp at the center have an odd number of neighbours that are on.

For the black lamps, i.e. $i + j$ even, we choose the values:

$$f(i, j) = \begin{cases} 0, \text{ if } \max(|i|, |j|) \equiv 0, 1 \mod 4, \\ 1, \text{ if } \max(|i|, |j|) \equiv 2, 3 \mod 4. \end{cases}$$

For the white lamps, i.e. $i+j$ odd, we choose the values:

$$f(i,j) = \begin{cases} 0, \text{ if } \max(|i|, |j|-1) \equiv 0,1 \mod 4, \\ 1, \text{ if } \max(|i|, |j|-1) \equiv 2,3 \mod 4. \end{cases}$$

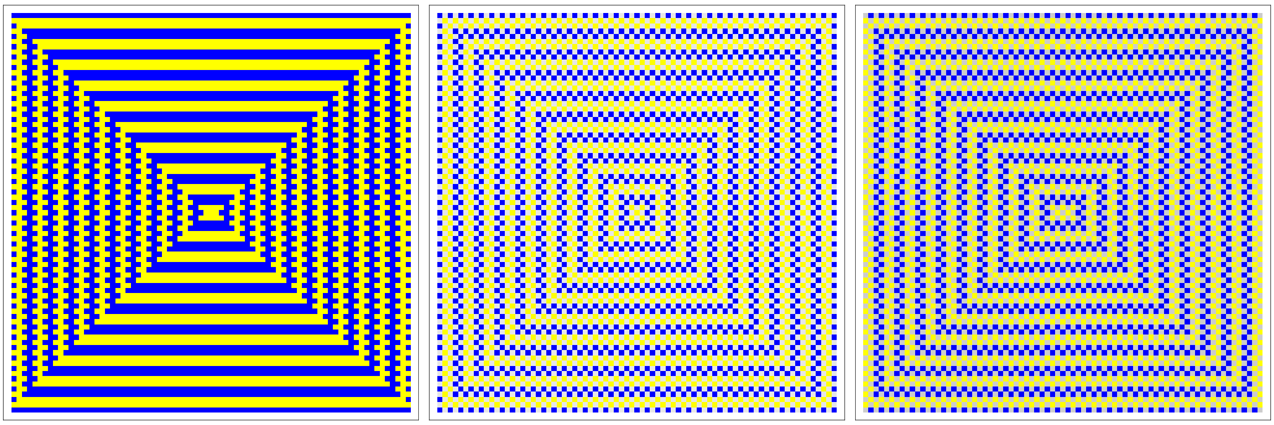
(This assignment can be found by replacing 2017 with a small number, say 17, starting with a row of zeros, using the assumptions to determine the rest and then notice that the zeros and ones for black or white lamps only form frames of depth 2 around the center.)

It is now easily checked that the condition is satisfied for all non-central lamps:

For a white lamp we assume without loss of generality $|i| < |j|$ (equality is impossible because they have different parity). Then, for the neighbours $(i \pm 1, j)$ and $(i, j \pm 1)$, the bigger coordinates are $|j-1|$, $|j|$, $|j|$ and $|j+1|$ and we can check easily that an odd number of them are $\equiv 2,3 \mod 4$.

For a black lamp with $j > 0$ or $j < 0$, we argue anlogously. If $j = 0$, then $i \neq 0$ for a non-central lamp, therefore the maximum is $|i|$ and wie have again the values $|i-1|$, $|i|$, $|i|$, $|i+1|$ to check which contain and odd number of values $\equiv 0,1 \mod 4$.

Therefore, we have found an arrangement with exactly one lamp with an even number of neighbours that are on as desired.



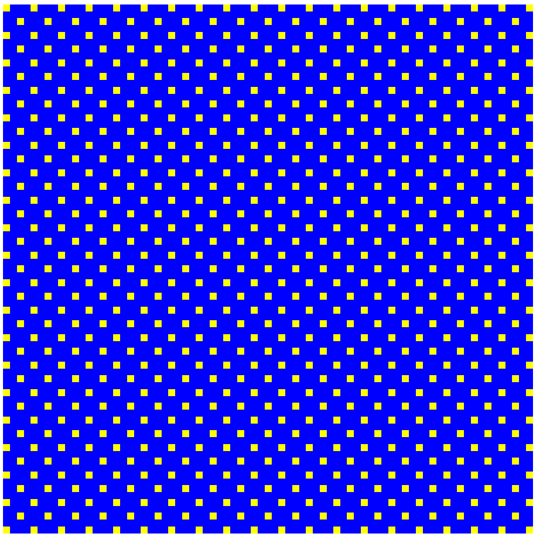
The images show the discussed optimal arrangement for $n = 77$. Lamps that are on are yellow, lamps that are off are blue. The first image shows all lamps, the second image shows the lamps with $i+j$ even and the third image shows the lamps with $i+k$ odd. ◀

Alternative solution. We color the board as a chess board in such a way that the four corners are white.

An active lamp on a black field has no influence on the number of active neighbours of any lamp on a black field, and vice versa an active lamp on a white field has no influence on the number of active neighbours of a lamp on a white field. Therefore, we can optimize the number of lamps with an even number of active neighbours separately for lamps on black and white fields.

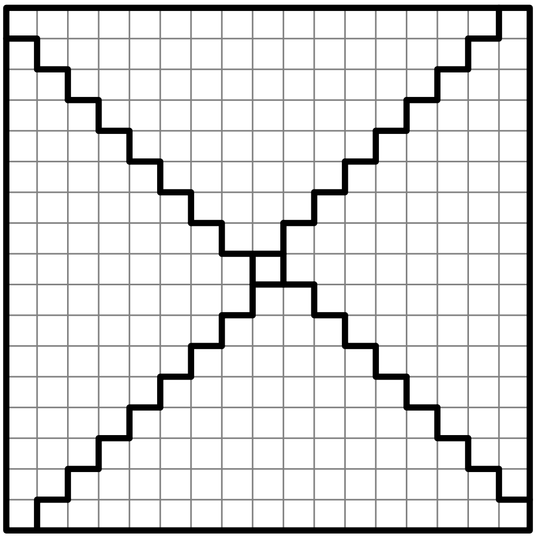
For the black fields, it is easy to find an arrangement in which all black fields have exactly one active neighbour, by turning on the following lamps: In the 1st, 5th, 9th, 13th, ... row the 1st, 5th, 9th, 13th, ... lamp (so all lamps with $x \equiv y \equiv 1 \mod 4$, assuming that the corner has coordinates $(1,1)$), and in the 3rd, 7th, 11th, ... row the 3rd, 7th, 11th, ... lamp (so all lamps with $x \equiv y \equiv 3 \mod 4$).

An example for $n = 77$ with the same color coding as in the previous solution:

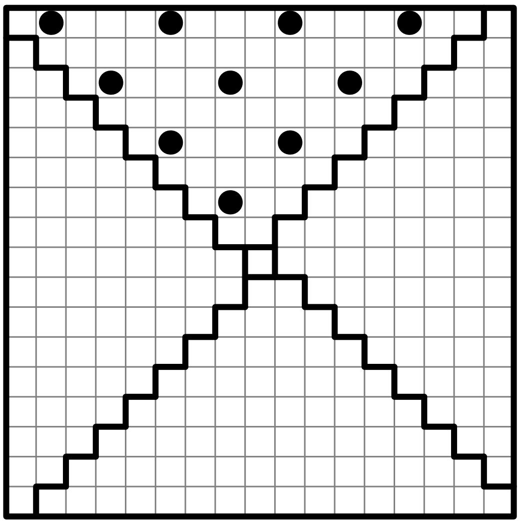


For white fields, we first show that at least one white field has to have an even number of active neighbours. To do so, we colour all white fields in the 1st, 3rd, 5th, ... row and in the 1st, 3rd, 5th, ... column red. Each black field is neighbour to exactly two red fields, and red fields have only black neighbours. If x lamps on black fields are active, then all red fields together have exactly $2x$ active neighbours. Since the number of red fields is odd, at least one of them has to have an even number of active neighbours.

All that is left to do is finding an arrangement in which all white fields except for one have an odd number of active neighbours. To do so, we separate the board into four "triangles" roughly as follows:

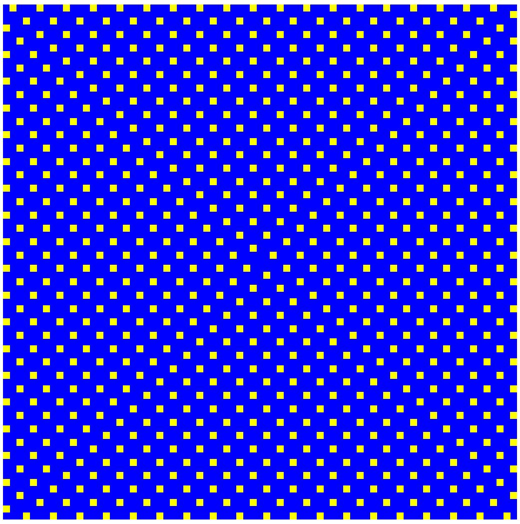


In the upper triangle we turn on the following lamps: In the 1st, 3rd, 5th, ... row always the lamp on the second field from the left, and then every 4th lamp, like this:

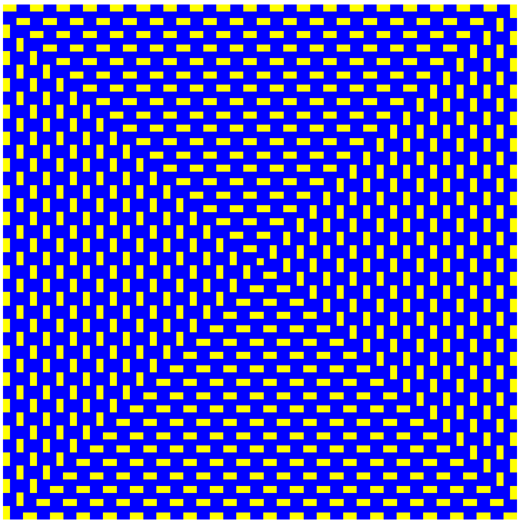


For filling the other triangles, we rotate the same pattern by 90°. Each lamp is only neighboured to fields from the own triangle, and each field within the triangle has exactly one active neighbour. Only the field in the middle of the board is left.

An example for $n = 77$:



Altogether, the pattern looks like this:



◀

T - 4

Let $n \geqslant 3$ be an integer. A sequence $P_1, P_2, \ldots, P_n$ of distinct points in the plane is called *good* if no three of them are collinear, the polyline $P_1P_2 \ldots P_n$ is non-self-intersecting and the triangle $P_iP_{i+1}P_{i+2}$ is oriented counterclockwise for every $i = 1, 2, \ldots, n-2$.

For every integer $n \geqslant 3$ determine the greatest possible integer k with the following property: there exist n distinct points $A_1, A_2, \ldots, A_n$ in the plane for which there are k distinct permutations $\sigma\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that $A_{\sigma(1)}, A_{\sigma(2)}, \ldots, A_{\sigma(n)}$ is good.

(A polyline $P_1P_2 \ldots P_n$ consists of the segments $P_1P_2, P_2P_3, \ldots, P_{n-1}P_n$.)

Answer: $n^2 - 4n + 6$.

Solution. Fix n points on a plane, no three of which are collinear. Let $\mathcal{P}$ be their convex hull. Let the vertices of $\mathcal{P}$ be $A_1, A_2, \ldots, A_m$ (lying in this order on the boundary of $\mathcal{P}$ counterclockwise). We denote $A_{m+1} = A_1$. Also, let $\mathcal{I}$ be the set of our fixed points other than $A_1, \ldots, A_m$, i.e. the points lying in the interior of $\mathcal{P}$.

Lemma. *Every good polyline contains all but one side of $\mathcal{P}$.*

Proof. The key observation is that if a segment A_iA_{i+1} is not a part of our polyline, then the point A_{i+1} appears in the polyline before A_i.

This is clear if A_i is the last vertex of the polyline. Otherwise there is a segment A_iX in the polyline, where $X \neq A_{i+1}$. Observe that all segments appearing after A_iX are located in the halfplane determined by the line A_iX which does not contain the point A_{i+1}. This is because the polyline always turns left, has no self-intersections, and A_i is a vertex of $\mathcal{P}$. This implies that the point A_{i+1} must appear in the polyline before A_i.

It is clear that at least one side of $\mathcal{P}$ does not appear in the polyline. Suppose now that $A_{i_1}A_{i_1+1}, A_{i_2}A_{i_2+1}, \ldots, A_{i_j}A_{i_j+1}$ are all segments on the boundary of $\mathcal{P}$ that do not appear in the polyline (where $1 \leqslant i_1 < i_2 < \ldots < i_j \leqslant n$ and $j \geqslant 2$). Using the observation we know that A_{i_1} appears after A_{i_1+1}, which is followed by $A_{i_1+2}, A_{i_1+3}, \ldots, A_{i_2}$. Since $A_{i_2} \neq A_{i_1}$, this means that A_{i_1} appears after A_{i_2}. Analogously, A_{i_2} appears after A_{i_3}, and so on, and A_{i_j} appears after A_{i_1}. Thus A_{i_1} appears after A_{i_1}, which is absurd. Therefore there is exactly one i such that A_iA_{i+1} does not belong to the polyline. ◀

Lemma. *For every good polyline there is a line which intersects exactly one segment of the polyline.*

Proof. Using the previous lemma we know that there is exactly one i such that A_iA_{i+1} does not belong to the polyline. Thus the polyline is of the form $B_1B_2 \ldots B_jA_{i+1}A_{i+2} \ldots A_{i-1}A_iC_1C_2 \ldots C_l$. (It may happen that there are no points before A_{i+1} and/or no points after A_i.)

It is quite clear that $B_1B_2 \ldots B_jA_{i+1}$ and $A_iC_1C_2 \ldots C_l$ can be separated by a line. Again, this follows form the fact that polyline has no self-intersections and always turns left, and the fact that you can separate two non-intersecting convex polygons by a line. It is clear that this separating line must intersect exactly one segment of the polyline $A_{i+1}A_{i+2}\dots A_{i-1}A_i$. ◀

Lemma. *Each good polyline is uniquely determined by an i such that A_iA_{i+1} is not in the polyline and by the partition $\mathcal{I} = \mathcal{B} \cup \mathcal{C}$ such that there is a line intersecting A_iA_{i+1} separating $\mathcal{B}$ from $\mathcal{C}$.*

Proof. This is easy to see. We use the previous lemma and the fact that the polyline only turns left and has no self-intersections. ◀

Lemma. *Consider the $\binom{n-m}{2}$ lines determined by the points of $\mathcal{I}$. Suppose that j of them intersect segment A_iA_{i+1}. Then there are exactly $n-m+j+1$ good polylines not containing A_iA_{i+1}.*

Proof. We will move a point X along the segment A_iA_{i+1}, starting from A_i, and count how many good partitions of $\mathcal{I}$ are there. In the beginning of our journey there are $n-m+1$ possible partitions of $\mathcal{I}$ by a line passing through X. Every time we cross a line determined by some two points of $\mathcal{I}$ we get exactly one new partition. Since we cross j such lines, the total number of good partitions is equal to $n-m+1+j$. This corresponds to $n-m+j+1$ good polylines. ◀

Lemma. *There are exactly $(n-m+1)m + 2\binom{n-m}{2}$ good polylines.*

Proof. For each i there are $n-m+j_i+1$ good polylines. Summing up yields

$$\sum_{i=1}^{m} n-m+j_i+1 = m(n-m+1) + \sum_{i=1}^{m} j_i = (n-m+1)m + 2\binom{n-m}{2}$$

because there are $\binom{n-m}{2}$ lines determined by points in $\mathcal{I}$ and each of them intersects two sides of $\mathcal{P}$. ◀

Since $m \mapsto 2\binom{n-m}{2} + (n-m+1)m$ is decreasing, it follows that the greatest possible number of good polylines is achieved for the smallest possible value of m, i.e. for $m = 3$. Therefore the answer is $2\binom{n-3}{2} + 3(n-2) = n^2 - 4n + 6$. ◀

T - 5

Let ABC be an acute-angled triangle with $AB > AC$ and circumcircle Γ. Let M be the midpoint of the shorter arc BC of Γ, and let D be the intersection of the rays AC and BM. Let $E \neq C$ be the intersection of the internal bisector of the angle ACB and the circumcircle of the triangle BDC. Let us assume that E is inside the triangle ABC and there is an intersection N of the line DE and the circle Γ such that E is the midpoint of the segment DN.

Show that N is the midpoint of the segment I_BI_C, where I_B and I_C are the excentres of ABC opposite to B and C, respectively.

Solution. Consider the following implications:

Let us denote by P the other point of intersection of internal bisector from C and the circle Γ.

$$\begin{aligned}
& BDCE \text{ is cyclic} \\
\Longrightarrow\ & \angle BDC = \angle BEP \\
\Longrightarrow\ & \triangle BEP \sim \triangle BDA && (\angle BPE = \angle BPC = \angle BAC = \angle BAD) \\
\Longrightarrow\ & \frac{BE}{BP} = \frac{BD}{BA} \\
\Longrightarrow\ & \frac{BE}{BD} = \frac{BP}{BA} \\
\Longrightarrow\ & \triangle BDE \sim \triangle BAP && (\angle DBE = \angle ABP = \frac{1}{2}\angle ACB) \\
\Longrightarrow\ & EB = ED && (\triangle BAP \text{ is isosceles}) \\
\Longrightarrow\ & \angle MBN = \angle DBN = 90^\circ \\
\Longrightarrow\ & M \text{ and } N \text{ are diametrically opposite on } \Gamma \\
\Longrightarrow\ & NB = NC.
\end{aligned}$$

The quadrilateral I_BI_CBC is cyclic, as $I_BB \perp I_CB$ and $I_CC \perp I_BC$. Let I_A be the excentre of $\triangle ABC$ opposite to A. Now consider the nine point circle of $\triangle I_AI_BI_C$. We can clearly see that this circle is Γ, as $\triangle ABC$ is the orthic triangle of $\triangle I_AI_BI_C$. So Γ passes through the midpoint of I_BI_C and this midpoint is also equidistant from B and C, so N must be the described midpoint. ◀

T - 6

Let ABC be an acute-angled triangle with $AB \neq AC$, circumcentre O and circumcircle Γ. Let the tangents to Γ through B and C meet each other at D, and let the line AO intersect BC at E. Denote the midpoint of BC by M and let AM meet Γ again at $N \neq A$. Finally, let $F \neq A$ be a point on Γ such that A, M, E and F are concyclic. Prove that FN bisects the segment MD.

Solution. We may suppose $AB < AC$. It is known that AD is a symmedian of triangle ABC. Let Q be its second point of intersection with k. The triangles ABM and and AQC are similar to each other, since their corresponding angles are equal, and it follows that

$$\angle AFQ = \angle ACQ = \angle AMB = \angle AFE.$$

Hence the points Q, E, and F are collinear.

Now let $R \neq F$ denote the point where FD intersects k again. As before, FD is a symmedian of triangle FBC, the triangles CFM and RFB are similar,

$$\angle EAF = \angle EMF = \angle RBF = \angle RAF,$$

and the points A, O, and R are collinear as well. So AR is a diameter of k and thus $\angle AFD = 90°$.

Now let X be the midpoint of AD and put $J = MD \cap FN$. As we have just seen, the triangle AXF is isosceles at X, the angle at its base being $\angle XAF = \angle QNJ$. But since AM is a median in triangle ABC and AQ the corresponding symmedian, we have $QN || BC \perp OD$, so the triangle QJN is likewise isosceles. As we have just seen, it has the same angle at its base. We thus have two similar isosceles triangles and looking at their remaining angles we learn $\angle FXA = \angle NJQ$. This implies that the quadrilateral $XQJF$ is cyclic.

Thus $\angle QXJ = \angle QFJ = \angle QFN = \angle QAN$ or in other words, the lines XJ and AM are parallel. Since X was defined to be the midpoint of AD, this tells us that J is the midpoint of MD, as desired. ◀

Alternative solution. As in the first solution we restrict our attention to the case $AB < AC$, define the points Q and J, and remark that the points Q, E, and F are collinear.

Our next step is to prove the similarity of the triangles JFM and JMN. Evidently their angles at J coincide and in view of

$$\angle MFJ = \angle MFQ + \angle QFN = \angle MAE + \angle QAM = \angle QAO$$

it remains to be shown that $\angle QAO = \angle JMN$. To this end we notice that because of $DM \cdot DO =$

$DB^2 = DQ \cdot DA$ the quadrilateral $AQMO$ is cyclic, and deduce that $\angle QAO = \angle OQA = \angle OMA = \angle JMN$ is indeed the case. This concludes the proof of $\triangle JFM \sim \triangle JMN$. Now $JM^2 = JN \cdot JF$ has become clear. Writing r for the radius of k and considering the power of J with respect to this circle we obtain

$$JN \cdot JF = JO^2 - r^2 = JO^2 - OM \cdot OD,$$

so altogether we have

$$OM \cdot OD = JO^2 - JM^2 = (JO - JM)(JO + JM) = OM \cdot (2JM + OM).$$

This simplifies immediately to $2JM = MD$, whereby the problem is solved. ◀

T - 7

Determine all integers $n \geqslant 2$ such that there exists a permutation $x_0, x_1, \ldots, x_{n-1}$ of the numbers $0, 1, \ldots, n-1$ with the property that the n numbers

$$x_0, \quad x_0 + x_1, \quad \ldots, \quad x_0 + x_1 + \ldots + x_{n-1}$$

are pairwise distinct modulo n.

Answer: All even numbers.

Solution. Suppose that $x_0, \ldots, x_{n-1}$ is such a permutation.

Note that $x_0 = 0$. Indeed, if $x_i = 0$ for some $i > 0$ then

$$x_0 + \cdots + x_{i-1} = x_0 + \cdots + x_{i-1} + x_i,$$

which is a contradiction.

On the other hand

$$x_0 + x_1 + \cdots + x_{n-1} = 0 + 1 + 2 + \cdots + n - 1 = n \cdot \frac{n-1}{2}.$$

This means that if n is odd then $x_0 + x_1 + \cdots + x_{n-1} \equiv 0 \pmod{n}$. This gives a contradiction if $n > 1$, because $x_0 = 0$.

If n is even then we put $x_i = i$ if i is even and $x_i = n - i$ if i is odd. Then

$$x_0 + x_1 + \cdots + x_{2m} = 0 + (n-1) + 2 + (n-3) + \cdots + 2m \equiv m \pmod{n}$$

and

$$x_0 + x_1 + \cdots + x_{2m+1} = x_0 + x_1 + \cdots + x_{2m} + (n - 2m - 1) \equiv n - m - 1 \pmod{n}.$$

Thus the numbers $x_0 + x_1 + \cdots + x_i$, $i = 0, 1, \ldots, n-1$, are pairwise distinct modulo n. ◀

T - 8

For an integer $n \geqslant 3$ we define the sequence $\alpha_1, \alpha_2, \ldots, \alpha_k$ as the sequence of exponents in the prime factor decomposition of $n! = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where $p_1 < p_2 < \cdots < p_k$ are primes.

Determine all integers $n \geqslant 3$ for which $\alpha_1, \alpha_2, \ldots, \alpha_k$ is a geometric progression.

Answer: The solutions are $n = 3, 4, 6, 10$.

Solution. Let p_i be the ith prime number and let α_i be the exponent of p_i in $n!$. It is well-known that

$$\alpha_i = \left\lfloor \frac{n}{p_i} \right\rfloor + \left\lfloor \frac{n}{p_i^2} \right\rfloor + \left\lfloor \frac{n}{p_i^3} \right\rfloor + \ldots .$$

For $n \geqslant 9$, we have

$$\begin{aligned} \alpha_2 &> \frac{n}{3} - \frac{2}{3} + \frac{n}{9} - \frac{8}{9} = \frac{4n-14}{9}, \\ \alpha_4 &> \frac{n-6}{7}, \\ \alpha_3 &< \frac{n}{5} + \frac{n}{25} + \cdots = \frac{n}{4}. \end{aligned}$$

In the geometric sequence, we have $\alpha_2 \alpha_4 = \alpha_3^2$. Since all terms in the above inequalities are positive, we get:

$$\left(\frac{4n-14}{9}\right)\left(\frac{n-6}{7}\right) < \alpha_2 \alpha_4 = (\alpha_3)^2 < \frac{n^2}{16}.$$

After simplification, we get: $n^2 - 608n + 1344 \leqslant 0$. This is certainly false for $n \geqslant 608$ and it remains to check the small cases.

For two primes p and q with $p < q < 2p$, we know that for $q \leqslant n \leqslant 2p - 1$, they will both have exponent 1 in the prime factor decomposition of $n!$ which is impossible because the ratio of the geometric sequence is bigger than 1 for $n > 3$. We will now give a list of appropriate primes p and q such that the intervals $[q, 2p-1]$ cover most of our interval.

p	3	5	7	11	17	29	47	83	157	311
q	5	7	11	13	19	31	53	89	163	313
$2p-1$	5	9	13	21	33	57	93	165	313	621

It remains to check $n = 3, 4, 6, 10$ which give the sequences of exponents $(1, 1)$, $(3, 1)$, $(4, 2, 1)$ and $(8, 4, 2, 1)$, respectively, which clearly work. ◀

Alternative solution. Let p_i be the ith prime number, that is $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc.

We check the small cases up to $n = 11$ and find the solutions $3!$, $4!$, $6!$ and $10!$.

From now on, let $n > 11$.

Claim. *The smallest and last exponent* α_k *in the prime factor decomposition of* $n!$ *is always* 1.

Proof. By Bertrand's postulate, we know $p_{k+1} < 2p_k$. Therefore, for all $n \in [p_k, p_{k+1} - 1]$, the largest prime number that occurs in $n!$ is p_k and it occurs exactly once. ◀

Therefore, $\alpha_1 = f^m$ for some $f \in \mathbb{N}$ and $m + 1$ is the number of primes in the prime factor decomposition of $n!$.

Case 1: $f = 2$.

The exponent of 2 in the prime factor decomposition of $(2^m)!$ is $2^{m-1}+2^{m-2}+\cdots+1 = 2^m-1$. Therefore, $\alpha_1 = 2^m$ holds exactly for $n = 2^m + 2$ and $2^m + 3$, and since $n > 11$, we have $m \geqslant 4$.

Let $\pi(x)$ be the number of primes $\leqslant x$. We check easily that $\pi(16) = 6$. Bertrand's postulate implies $\pi(2x) \geqslant \pi(x) + 1$, therefore,

$$\pi(n) \geqslant \pi(2^m) \geqslant \pi(2^{m-1}) + 1 \geqslant \ldots \geqslant \pi(2^4) + m - 4 = 6 + m - 4 = m + 2$$

which is not compatible with the fact that there are $m + 1$ primes in the prime factor decomposition of $n!$.

Case 2: $f > 2$.

Since the exponent of 2 is now even bigger than 2^m, n must be bigger than $2^m + 3$, so as before, the number of primes in the prime factor decomposition of $n!$ must be bigger than $m+1$ which is again a contradiction.

We have seen, that the solutions $3, 4, 6, 10$ are indeed all the solutions. ◀