The 12th Middle European Mathematical Olympiad

Problems with Solutions

The Problem Selection Committee

The Jury and the Problem Selection Committee selected 12 problems proposed by the following countries:

I-1	Austria
I-2	Ukraine
I-3	Slovakia
I-4	Slovakia

T-1	Poland
T-2	Austria
T-3	Czech Republic
T-4	Poland
T-5	Slovakia
T-6	Ukraine
T-7	Poland
T-8	Germany

The Problem Selection Committee would like to thank **Roger Labahn** for providing the LaTeX templates.

Contents

I-1

Let $\mathbb{Q}^+$ denote the set of all positive rational numbers and let $\alpha \in \mathbb{Q}^+$. Determine all functions $f\colon \mathbb{Q}^+ \to (\alpha, +\infty)$ satisfying

$$f\left(\frac{x+y}{\alpha}\right) = \frac{f(x)+f(y)}{\alpha}, \qquad \text{for all } x, y \in \mathbb{Q}^+.$$

I-2

The two figures depicted below consisting of 6 and 10 unit squares, respectively, are called *staircases*.

Consider a 2018×2018 board consisting of 2018^2 cells, each being a unit square. Two arbitrary cells were removed from the same row of the board. Prove that the rest of the board cannot be cut (along the cell borders) into staircases (possibly rotated).

I-3

Let ABC be an acute-angled triangle with $AB < AC$, and let D be the foot of its altitude from A. Let R and Q be the centroids of the triangles ABD and ACD, respectively. Let P be a point on the line segment BC such that $P \neq D$ and the points P, Q, R and D are concyclic. Prove that the lines AP, BQ and CR are concurrent.

I-4

(a) Prove that for every positive integer m there exists an integer $n \geq m$ such that

$$\left\lfloor \frac{n}{1} \right\rfloor \cdot \left\lfloor \frac{n}{2} \right\rfloor \cdots \left\lfloor \frac{n}{m} \right\rfloor = \binom{n}{m}. \tag{$*$}$$

(b) Denote by $p(m)$ the smallest integer $n \geq m$ such that the equation $(*)$ holds. Prove that $p(2018) = p(2019)$.

Remark: For a real number x, we denote by $\lfloor x \rfloor$ the largest integer not larger than x.

T-1

Let a, b and c be positive real numbers satisfying $abc = 1$. Prove that

$$\frac{a^2-b^2}{a+bc}+\frac{b^2-c^2}{b+ca}+\frac{c^2-a^2}{c+ab} \leq a+b+c-3.$$

T-2

Let $P(x)$ be a polynomial of degree $n \geq 2$ with rational coefficients such that $P(x)$ has n pairwise different real roots forming an arithmetic progression. Prove that among the roots of $P(x)$ there are two that are also the roots of some polynomial of degree 2 with rational coefficients.

T-3

A group of pirates had an argument and now each of them holds some other two at gunpoint. All the pirates are called one by one in some order. If the called pirate is still alive, he shoots both pirates he is aiming at (some of whom might already be dead). All shots are immediately lethal. After all the pirates have been called, it turns out that exactly 28 pirates got killed.

Prove that if the pirates were called in whatever other order, at least 10 pirates would have been killed anyway.

T-4

Let n be a positive integer and $u_1, u_2, \dots, u_n$ be positive integers not larger than 2^k, for some integer $k \geq 3$. A *representation* of a non-negative integer t is a sequence of non-negative integers $a_1, a_2, \dots, a_n$ such that

$$t = a_1u_1 + a_2u_2 + \cdots + a_nu_n.$$

Prove that if a non-negative integer t has a representation, then it also has a representation where less than $2k$ of the numbers $a_1, a_2, \dots, a_n$ are non-zero.

T-5

Let ABC be an acute-angled triangle with $AB < AC$, and let D be the foot of its altitude from A. Points B' and C' lie on the rays AB and AC, respectively, so that points B', C' and D are collinear and points B, C, B' and C' lie on one circle with center O. Prove that if M is the midpoint of BC and H is the orthocenter of ABC, then $DHMO$ is a parallelogram.

T-6

Let ABC be a triangle. The internal bisector of $\angle ABC$ intersects the side AC at L and the circumcircle of triangle ABC again at $W \neq B$. Let K be the perpendicular projection of L onto AW. The circumcircle of triangle BLC intersects line CK again at $P \neq C$. Lines BP and AW meet at point T. Prove that $AW = WT$.

T-7

Let $a_1, a_2, a_3, \ldots$ be the sequence of positive integers such that

$$a_1 = 1 \qquad \text{and} \qquad a_{k+1} = a_k^3 + 1, \text{ for all positive integers } k.$$

Prove that for every prime number p of the form $3\ell + 2$, where ℓ is a non-negative integer, there exists a positive integer n such that a_n is divisible by p.

T-8

An integer n is called *Silesian* if there exist positive integers a, b and c such that

$$n = \frac{a^2 + b^2 + c^2}{ab + bc + ca}.$$

(a) Prove that there are infinitely many Silesian integers.

(b) Prove that not every positive integer is Silesian.

I-1

Let $\mathbb{Q}^+$ denote the set of all positive rational numbers and let $\alpha \in \mathbb{Q}^+$. Determine all functions $f\colon \mathbb{Q}^+ \to (\alpha, +\infty)$ satisfying

$$f\left(\frac{x+y}{\alpha}\right) = \frac{f(x)+f(y)}{\alpha}, \qquad \text{for all } x, y \in \mathbb{Q}^+.$$

Answer. For $\alpha = 2$ the solutions of our functional equation are given by $f(x) = Ax + B$ for all $x \in \mathbb{Q}^+$, where either $A > 0$ and $B \geq 2$ or $A = 0$ and $B > 2$. For $\alpha \neq 2$ there are no solutions.

Solution. By putting $x = y$ in the given functional equation we get $f\left(\frac{2x}{\alpha}\right) = f(x) \cdot \frac{2}{\alpha}$. It follows that

$$t \in \mathrm{Im}(f) \iff t \cdot \frac{2}{\alpha} \in \mathrm{Im}(f) \qquad \text{for all } t \in \mathbb{Q}^+.$$

Therefore, if $\alpha \neq 2$ then f takes arbitrarily small values. This is a contradiction with the assumption that $f(x) > \alpha$ for all $x \in \mathbb{Q}^+$. We conclude that there are no such functions for $\alpha \neq 2$.

Assume now that $\alpha = 2$. By putting $x = a + b$ and $y = a - b$ in the given functional equation, where $a > b > 0$ are any rationals, we get

$$f(a+b) - f(a) = f(a) - f(a-b).$$

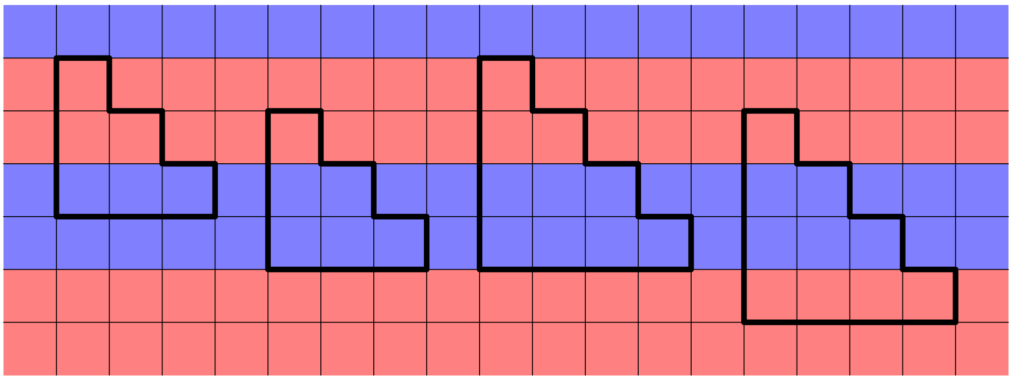
It follows that f restricted to any arithmetic sequence is linear. Since for every rational number q there is an arithmetic sequence containing q, 1, and 2, it follows that f is linear on $\mathbb{Q}^+$. Therefore $f(x) = Ax + B$ for some reals A and B. A direct check of the condition $f(x) > 2$ for all $x \in \mathbb{Q}^+$ yields that it must be that either $A > 0$ and $B \geq 2$ or $A = 0$ and $B > 2$. Clearly, all such functions satisfy the given equation.

I-2

The two figures depicted below consisting of 6 and 10 unit squares, respectively, are called *staircases*.

Consider a 2018×2018 board consisting of 2018^2 cells, each being a unit square. Two arbitrary cells were removed from the same row of the board. Prove that the rest of the board cannot be cut (along the cell borders) into staircases (possibly rotated).

Solution. Enumerate the rows of the board with integers from 1 to 2018. We color the cells of the board in horizontal strips of width 2 as follows: rows 1 and 2 are colored red, rows 3 and 4 are colored blue, rows 5 and 6 are colored red, rows 7 and 8 are colored blue, etc. If we disregarded the two cells removed from the board, both the number of red cells and the number of blue cells would be divisible by 4. Since the two cells are removed from the same row, they would have the same color, hence after the removal we have that either the number of red cells is divisible by 4, while the number of blue cells is congruent to 2 modulo 4, or vice versa. In both cases, the numbers of red cells and of blue cells are not congruent modulo 4.



It now remains to observe that if a staircase, either of size 6 or 10, is placed on the board, then the difference of the numbers of red and blue cells covered by the staircase is always divisible by 4. This follows from a straightforward case study. Hence, if the board with the two cells removed could be tiled with staircases, then the difference of the numbers of red and blue cells would be divisible by 4, a contradiction.

I-3

Let ABC be an acute-angled triangle with $AB < AC$, and let D be the foot of its altitude from A. Let R and Q be the centroids of the triangles ABD and ACD, respectively. Let P be a point on the line segment BC such that $P \neq D$ and the points P, Q, R and D are concyclic. Prove that the lines AP, BQ and CR are concurrent.

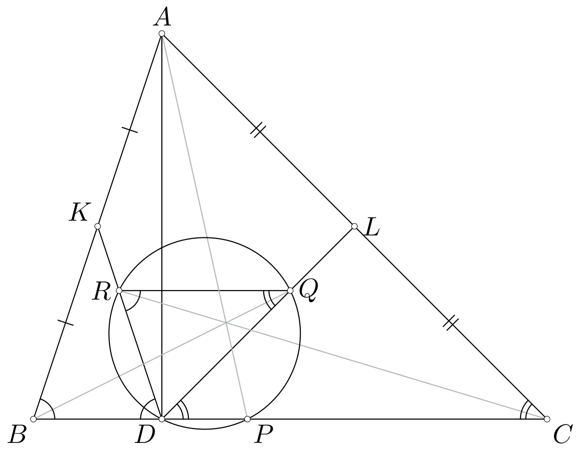
Solution 1. Without loss of generality, we may assume that P lies on the line segment CD. Let K, L be the midpoints of the sides AB, AC, respectively. Obviously

$$\angle CBA = \angle KDB = \angle RQP \quad \text{and} \quad \angle ACB = \angle CDL = \angle PRQ. \tag{*}$$

Furthermore, from the fact that

$$\frac{DR}{DK} = \frac{2}{3} = \frac{DQ}{DL}$$

we can see that $RQ \parallel KL$. Since $BC \parallel KL$ we also have $RQ \parallel BC$. This, together with angle equalities (*), implies that the sides PQ, QR, RP of triangle PQR are parallel to the sides AB, BC, CA of triangle ABC, respectively. Obviously those triangles are not congruent, which means that there exists a homothety which maps triangle PQR to ABC. The center of this homothety is the common intersection point of the lines AP, BQ, CR.



Solution 2. The fact that triangles ABC, PQR are homothetic may be proven in a slightly different way, as follows.

Let M be the midpoint of BC. Since the points A, D are symmetric with respect to the line KL, it follows that

$$\angle KML = \angle BAC = \angle KDL = \angle RPQ.$$

Consider a homothety h with center D and ratio $\frac{DK}{DR} = \frac{3}{2} = \frac{DL}{DQ}$. Then h maps the line segment QR to the line segment LK. Furthermore, h maps the point P to the point P' lying on the line BC such that $\angle LP'K = \angle QPR$. But there are exactly two points that satisfy those conditions, namely D and M. Since $P' \neq D$ we have $P' = M$ and therefore h maps triangle PQR to MLK. Composing h and the homothety centered at the centroid of ABC with ratio -2 which maps MLK to ABC, we obtain a homothety with negative ratio which maps PQR to ABC.

I-4

(a) Prove that for every positive integer m there exists an integer $n \geq m$ such that

$$\left\lfloor \frac{n}{1} \right\rfloor \cdot \left\lfloor \frac{n}{2} \right\rfloor \cdots \left\lfloor \frac{n}{m} \right\rfloor = \binom{n}{m}. \tag{$*$}$$

(b) Denote by $p(m)$ the smallest integer $n \geq m$ such that the equation $(*)$ holds. Prove that $p(2018) = p(2019)$.

Remark: For a real number x, we denote by $\lfloor x \rfloor$ the largest integer not larger than x.

Solution. It is clear that $p(1) = 1$ and $p(2) = 3$. From now on we assume that $m \geq 3$.

First, we prove that for all positive integers n and k with $1 \leq k \leq n$, it holds that

$$\left\lfloor \frac{n}{k} \right\rfloor \geq \frac{n-k+1}{k}.$$

Indeed, if we write n as $ik + r$ where $0 \leq r \leq k-1$, then

$$\left\lfloor \frac{n}{k} \right\rfloor = i \geq \frac{ik + r - (k-1)}{k} = \frac{n-k+1}{k}.$$

Note here that the equality holds if and only if $r = k-1$, that is, $n+1$ is divisible by k.

Therefore, for all $n \geq m$ we have

$$\left\lfloor \frac{n}{1} \right\rfloor \cdot \left\lfloor \frac{n}{2} \right\rfloor \cdots \left\lfloor \frac{n}{m} \right\rfloor \geq \frac{n}{1} \cdot \frac{n-1}{2} \cdot \ldots \cdot \frac{n-m+1}{m} = \binom{n}{m}.$$

The equality holds if and only if k divides $n+1$ for all $1 \leq k \leq m$. Since $m \geq 3$, we have $\operatorname{lcm}(1, 2, \ldots, m) > m$. Thus the least n that satisfies these conditions is $\operatorname{lcm}(1, 2, \ldots, m) - 1$. It follows that $p(m) = \operatorname{lcm}(1, 2, \ldots, m) - 1$ for every integer $m \geq 3$. In particular, a number n as in statement a) always exists.

To finish the problem we need to prove that $\operatorname{lcm}(1, 2, \ldots, 2018) = \operatorname{lcm}(1, 2, \ldots, 2019)$. But this is clear, because $2019 = 3 \cdot 673 \mid \operatorname{lcm}(1, 2, \ldots, 2018)$, so we are done.

T-1

Let a, b and c be positive real numbers satisfying $abc = 1$. Prove that

$$\frac{a^2-b^2}{a+bc}+\frac{b^2-c^2}{b+ca}+\frac{c^2-a^2}{c+ab} \leq a+b+c-3.$$

Solution. Note that

$$\frac{a^2-b^2}{a+bc}=\frac{a(a+bc)-abc-b^2}{a+bc}=a-\frac{b^2+1}{a+bc}=a-a\cdot\frac{b^2+1}{a^2+abc}=a-a\cdot\frac{b^2+1}{a^2+1}.$$

Therefore the desired inequality can be rewritten as

$$a-a\cdot\frac{b^2+1}{a^2+1}+b-b\cdot\frac{c^2+1}{b^2+1}+c-c\cdot\frac{a^2+1}{c^2+1}\leq a+b+c-3,$$

or

$$3 \leq a\cdot\frac{b^2+1}{a^2+1}+b\cdot\frac{c^2+1}{b^2+1}+c\cdot\frac{a^2+1}{c^2+1}.$$

This immediately follows from AM-GM inequality. The proof is completed.

T-2

Let $P(x)$ be a polynomial of degree $n \geq 2$ with rational coefficients such that $P(x)$ has n pairwise different real roots forming an arithmetic progression. Prove that among the roots of $P(x)$ there are two that are also the roots of some polynomial of degree 2 with rational coefficients.

Solution. Let

$$P(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0.$$

Then the polynomial P has n distinct real roots $r_1 < r_2 < \ldots < r_n$ forming an arithmetic progression. By Viète's formula, we have $r_1 + r_2 + \ldots + r_n = -a_{n-1}$. Since a_{n-1} is rational, we infer that $\mu = \frac{r_1+\ldots+r_n}{n}$ is rational as well.

Consider now polynomial $Q(x) = P(x+\mu)$. Since μ is rational, Q also has rational coefficients. Moreover, Q has roots $r'_1 < r'_2 < \ldots < r'_n$ with $r'_i = r_i - \mu$, which also form an arithmetic progression. We now consider two cases: either n is even or n is odd.

Case 1: n is odd. Since the mean of $r'_1, r'_2, \ldots, r'_n$ is equal to 0, we can write

$$(r'_1, r'_2, \ldots, r'_n) = \left(-\frac{n-1}{2} \cdot p, -\frac{n-3}{2} \cdot p, \ldots, -p, 0, p, \ldots, \frac{n-3}{2} \cdot p, \frac{n-1}{2} \cdot p\right)$$

for some real $p > 0$. Denoting $k = \frac{n-1}{2}$, again by Viète's formula we infer that

$$a_{n-2} = \sum_{-k \leq i < j \leq k} (ip) \cdot (jp) = p^2 \cdot \sum_{-k \leq i < j \leq k} ij.$$

Observe that for all $1 \leq a < b \leq k$, summands $(-a) \cdot b$, $(-b) \cdot a$, $(-b) \cdot (-a)$ and $a \cdot b$ in the sum $\sum_{-k \leq i < j \leq k} ij$ cancel out, and the only summands that do not cancel out are of the form $(-a) \cdot a$ for $1 \leq a \leq k$. Hence

$$\sum_{-k \leq i < j \leq k} ij = -\sum_{1 \leq i \leq k} i^2.$$

Since $\sum_{1 \leq i \leq k} i^2$ is a positive integer, we have

$$p^2 = \frac{-a_{n-2}}{\sum_{1 \leq i \leq k} i^2},$$

hence p^2 is rational.

Consider now the quadratic polynomial $S(x) = (x-p)(x+p) = x^2 - p^2$. Observe that S has rational coefficients and both its roots are also distinct roots of Q. Hence, the polynomial $T(x) = S(x-\mu)$ is quadratic, has rational coefficients with the leading coefficient equal to 1, and its roots are $-p + \mu = r_{\frac{n-1}{2}}$ and $p + \mu = r_{\frac{n+3}{2}}$.

Case 2: n is even. Again, this means that we can write

$$(r'_1, r'_2, \ldots, r'_n) = (-(n-1)\cdot p, -(n-3)\cdot p, \ldots, -3p, -p, p, 3p, \ldots, (n-3)\cdot p, (n-1)\cdot p)$$

for some real $p > 0$. Denoting $k = \frac{n}{2}$, by Viète's formula we infer that

$$a_{n-2} = \sum_{-k<i<j\leq k} (2i-1)p \cdot (2j-1)p = p^2 \cdot \sum_{-k<i<j\leq k} (2i-1)(2j-1).$$

A similar cancelling scheme as in Case 1 yields that

$$\sum_{-k<i<j\leq k} (2i-1)(2j-1) = -\sum_{1\leq i\leq k} (2i-1)^2.$$

Again, $\sum_{1\leq i\leq k}(2i-1)^2$ is a positive integer, hence

$$p^2 = \frac{-a_{n-2}}{\sum_{1\leq i\leq k}(2i-1)^2}$$

is a rational. We may again consider the quadratic polynomial $T(x) = S(x-\mu)$ where $S(x) = (x-p)(x+p) = x^2 - p^2$, and conclude as in Case 1.

T-3

A group of pirates had an argument and now each of them holds some other two at gunpoint. All the pirates are called one by one in some order. If the called pirate is still alive, he shoots both pirates he is aiming at (some of whom might already be dead). All shots are immediately lethal. After all the pirates have been called, it turns out that exactly 28 pirates got killed.

Prove that if the pirates were called in whatever other order, at least 10 pirates would have been killed anyway.

Solution. Call a pirate *mortal* if someone is aiming at him in the beginning. Since some order of shooting results in 28 pirates dead, there are at least 28 mortal pirates.

For the sake of contradiction, suppose that some order of shooting results in at most 9 dead pirates. Then at least 19 mortal pirates survive. Each mortal pirate is pointed at by some other pirate but the (at most) 9 dead pirates cannot point at more than 18 different pirates. Hence there is a mortal pirate, call him Will, who is pointed at by another pirate, call him Jack, and both Will and Jack survive till the end of the shooting. But then at some point Jack is called and he kills Will, a contradiction.

T-4

Let n be a positive integer and $u_1, u_2, \ldots, u_n$ be positive integers not larger than 2^k, for some integer $k \geq 3$. A *representation* of a non-negative integer t is a sequence of non-negative integers $a_1, a_2, \ldots, a_n$ such that

$$t = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n.$$

Prove that if a non-negative integer t has a representation, then it also has a representation where less than $2k$ of the numbers $a_1, a_2, \ldots, a_n$ are non-zero.

Solution. We shall treat a representation of t as a multiset with entries from $\{u_1, \ldots, u_n\}$ whose sum is equal to t, where integers $a_1, \ldots, a_n$ correspond to the multiplicities of numbers $u_1, \ldots, u_n$ in the multiset. For a representation M, let the *support* of M be the subset of those elements of $\{u_1, \ldots, u_n\}$ that appear at least once in M. The problem boils down to proving that if there is some representation of t, then there is also one with support of size less than $2k$.

Ordering numbers $u_1, \ldots, u_n$ as in this sequence, let M be the lexicographically smallest representation of t. We shall prove that M satisfies the claim. Suppose for contradiction that the support of M has size at least $2k$. Let S be any subset of the support of M consististing of $2k$ numbers and for each $X \subseteq S$, consider the sum of numbers in X. This sum is an integer between 0 and $2k \cdot 2^k$, for which there are $1 + 2k \cdot 2^k$ possibilities. On the other hand, there are 2^{2k} different subsets of S. Since $1 + 2k \cdot 2^k < 2^{2k}$ for $k \geq 3$ (straightforward induction), we infer that there are two different subsets X, Y of S such that X and Y have the same sum.

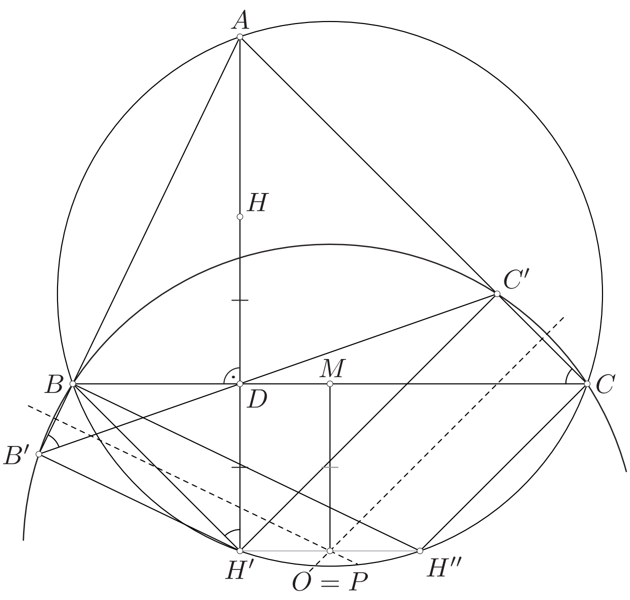
Since both X and Y are subsets of S, they are also sub(multi)sets of M. Consider constructing two multisets M', M'' from M: M' is obtained by subtracting Y and adding X, whereas M'' is obtained by subtracting X and adding Y. Since the sums of X and Y are equal, it follows that M' and M'' are both representations of t. However, either M' or M'' is lexicographically smaller than M, depending on whether X is lexicographically smaller than Y. This is a contradiction with the choice of M.

T-5

Let ABC be an acute-angled triangle with $AB < AC$, and let D be the foot of its altitude from A. Points B' and C' lie on the rays AB and AC, respectively, so that points B', C' and D are collinear and points B, C, B' and C' lie on one circle with center O. Prove that if M is the midpoint of BC and H is the orthocenter of ABC, then $DHMO$ is a parallelogram.

Solution 1. Without loss of generality assume that $AB < AC$. Let H', H'' be the points symmetric to H with respect to BC and with respect to M, respectively. It is well-known that H', H'' both lie on the circumcircle of ABC; furthermore, AH'' is the diameter of this circumcircle. Since $DH \parallel MO$ (they are both perpendicular to BC), it suffices to prove that $MO = DH$. We will prove that O is the midpoint of $H'H''$ which leads easily to the conclusion.

Since $\angle BH'A = \angle BCA = \angle BB'D$, we see that the points B, B', H', D are concyclic. Therefore $\angle H'B'B = 180° - \angle BDH' = 90°$ and we find that $BB'H'H''$ is a right-angled trapezoid. It follows that the midpoint P of $H'H''$ lies on the perpendicular bisector of BB'. Analogously we can prove that P lies on the perpendicular bisector of CC'. However, BB' and CC' are chords of a circle with center O, which means that the intersection of their perpendicular bisectors is O. This implies that $O = P$ and finishes the proof.



Solution 2. As in the first solution we introduce the point H' and we observe that it lies on the circumcenter of ABC. By the power of point theorem we get

$$DA \cdot DH' = DB \cdot DC = DB' \cdot DC'$$

and therefore B', H', C', A are concyclic. We conclude that

$$\angle AH'C' = \angle AB'C' = \angle BCA.$$

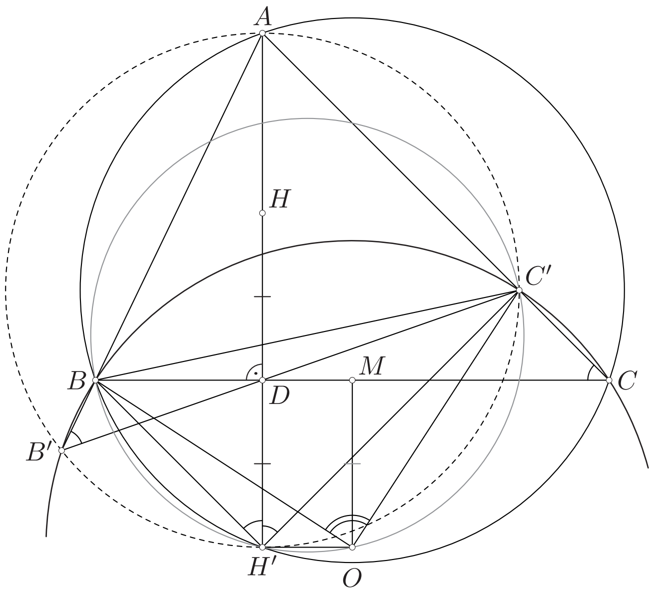
Thus

$$\angle BH'C' = \angle BH'A + \angle AH'C' = 2\angle BCA = \angle BOC'$$

which leads us to the conclusion that B, H', O, C' are concyclic. Since the triangle BOC' is isosceles, $\angle C'BO = 90° - \frac{1}{2}\angle BOC' = 90° - \angle BCA$ and therefore

$$\angle DH'O = \angle AH'C' + \angle C'H'O = \angle BCA + \angle C'BO = \angle BCA + 90° - \angle BCA = 90°.$$

From $AD \perp BC$ and $OM \perp BC$ we deduce that $DMOH'$ is a rectangle which implies that $OM = H'D = DH$. Together with the fact that lines DH, MO are parallel (they are both perpendicular to BC), this leads us to the final conclusion.



T-6

Let ABC be a triangle. The internal bisector of $\angle ABC$ intersects the side AC at L and the circumcircle of triangle ABC again at $W \neq B$. Let K be the perpendicular projection of L onto AW. The circumcircle of triangle BLC intersects line CK again at $P \neq C$. Lines BP and AW meet at point T. Prove that $AW = WT$.

Solution 1. Note that

$$\angle BPC = \angle BLC = \angle BAC + \frac{\angle CBA}{2} = \angle BAW.$$

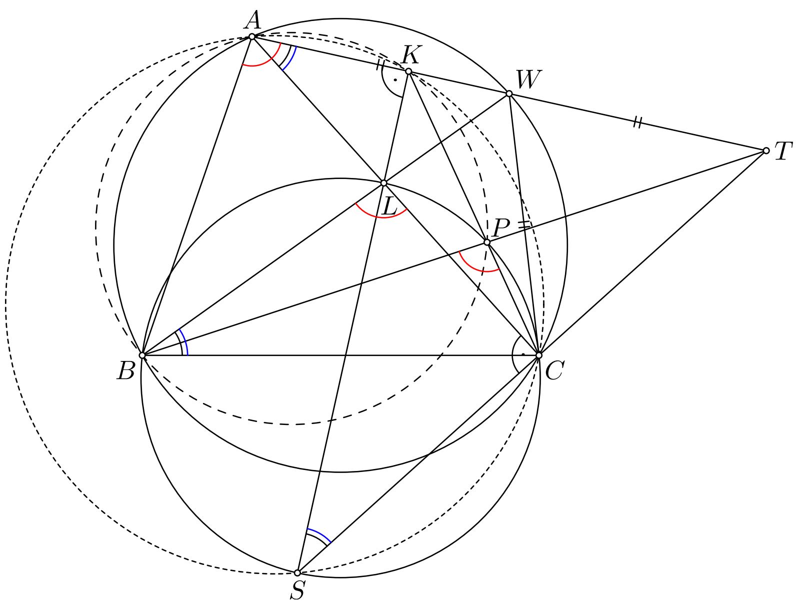
Therefore, $ABPK$ is cyclic.

Let KL intersect the circumcircle of triangle BLC at a point $S \neq L$. Then

$$\angle KSC = \angle LSC = \angle LBC = \angle WBC = \angle WAC = \angle KAC.$$

So $KASC$ is cyclic.

Now, radical axes of circles $KABP, BPCS, KASC$ are concurrent. That is: AK, BP, CS concur and so CS passes through T. Note that $\angle SCA = \angle SKA = 90°$, so $\angle TCA = 90°$. As $AW = WC$, we find that W is the circumcenter of the (right) triangle ACT, so $AW = WT$.

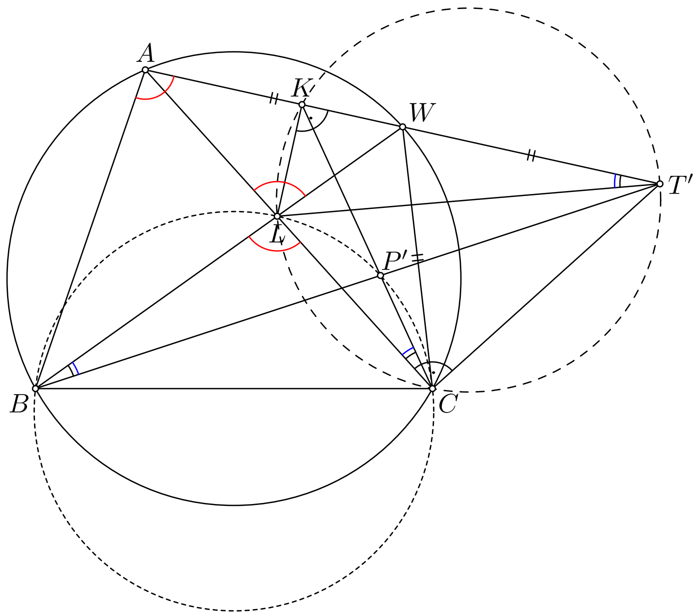


Solution 2. Let T' be the point on the ray AW such that $AW = WT'$. We will show that $T' = T$.

As $WA = WC = WT'$, C lies on the circle with diameter AT', so $\angle ACT' = 90^\circ = \angle LKT'$. It follows that $KLCT'$ is cyclic.

Note that $\angle ALW = \angle BLC = \angle BAC + \frac{\angle CBA}{2} = \angle BAW$. Therefore, triangles AWL and BWA are similar. It follows that $WA^2 = WL \cdot WB$. So $WT'^2 = WL \cdot WB$, and triangles $T'WL$ and BWT' are similar.

Now, let BT' intersect KC at P'. We will show that $P' = P$, from which we will get $T' = T$. Note that from $\triangle T'WL \sim \triangle BWT'$ we get $\angle LT'W = \angle LBT' = \angle LBP'$. As $KLCT'$ is cyclic, $\angle LT'W = \angle LT'K = \angle LCK = \angle LCP'$. Combining, we get $\angle LCP' = LBP'$, meaning that $LBCP'$ is cyclic. As $LBCP$ is cyclic, $P' = P$, as desired.



T-7

Let $a_1, a_2, a_3, \ldots$ be the sequence of positive integers such that

$$a_1 = 1 \qquad \text{and} \qquad a_{k+1} = a_k^3 + 1, \text{ for all positive integers } k.$$

Prove that for every prime number p of the form $3\ell + 2$, where ℓ is a non-negative integer, there exists a positive integer n such that a_n is divisible by p.

Solution. Let $f(x) = x^3 + 1$. In what follows all congruences are considered modulo p.

We will prove that if $f(a) \equiv f(b)$ for some integers a and b, then $a \equiv b$. This is clear when $f(a) \equiv 1$, because then $a \equiv b \equiv 0$. Otherwise $a \not\equiv 0 \not\equiv b$. By Fermat's little theorem, we have $a^{3k+1} \equiv 1 \equiv b^{3k+1}$ and hence

$$f(a) \equiv f(b) \implies a^3 \equiv b^3 \implies (a^3)^{2k+1} \equiv (b^3)^{2k+1} \implies a \cdot (a^{3k+1})^2 \equiv b \cdot (b^{3k+1})^2 \implies a \equiv b.$$

The above observation implies that f considered as a map from $\mathbb{F}_p$ to $\mathbb{F}_p$ is an injection, and therefore it is a permutation, because $\mathbb{F}_p$ is finite. Let ℓ be the length of the cycle of this permutation that contains 1. Then

$$f(f^{\ell-1}(1)) \equiv f^{\ell}(1) \equiv 1 \equiv f(0).$$

Since f is injective, it follows that $f^{\ell-1}(1) \equiv 0$. Therefore $n = \ell - 1$ is as required.

Comment. The fact that f is a permutation on $\mathbb{F}_p$ for p of the form $3k+2$ can be also proved as follows. We have

$$f(a) - f(b) = (a-b)(a^2 + ab + b^2) = (a-b)b^2((ab^{-1})^2 + (ab^{-1}) + 1).$$

Therefore, to prove that $f(a) \equiv f(b)$ entails $a \equiv b$ it suffices to show that the polynomial $t^2 + t + 1$ has no roots in $\mathbb{F}_p$. The discriminant of this polynomial is -3 and one can readily verify using Legendre's symbol that -3 is not a quadratic residue modulo p if and only if p is of the form $3k+2$ for an integer k.

T-8

An integer n is called *Silesian* if there exist positive integers a, b and c such that

$$n = \frac{a^2+b^2+c^2}{ab+bc+ca}.$$

(a) Prove that there are infinitely many Silesian integers.

(b) Prove that not every positive integer is Silesian.

Solution 1. to a) First, we try to find k such that $k = \dfrac{a^2+b^2+c^2}{ab+bc+ca}$ for some (not necessarily positive) integers a, b, c. In order to reduce the number of variables, we look for solutions satisfying $a+b=1$. Substituting $b = 1-a$ we find

$$k = \frac{2a^2-2a+1+c^2}{c+a(1-a)} = \frac{(c+1)^2}{c+a(1-a)} - 2.$$

We take $c = 1-a(1-a) = a^2-a+1$ so that the denominator is equal to 1. This forces $k = (c+1)^2 - 2 = (a^2-a+2)^2 - 2$.

It follows that if $c = a^2-a+1$ and $k = (a^2-a+2)^2-2$ then $b_1 = 1-a$ is a root of the following quadratic equation in variable b:

$$a^2+b^2+c^2 = k(ab+bc+ca).$$

The other root is, by Viète's formula, $b_2 = k(a+c) - b_1 = k(a+c)+a-1$.

It is clear that for every integer $a > 0$ one has $c = a^2-a+1 > 0$, $k = (a^2-a+2)^2-2 > 0$, and $b = k(a+c)+a-1 > 0$. These numbers satisfy

$$\frac{a^2+b^2+c^2}{ab+bc+ca} = k$$

and witness that $k = (a^2-a+2)^2-2 \in S$ for every positive integer a.

Solution 2. to a) To prove that $k \in S$ we have to find positive integers a, b, c satisfying

$$k \cdot (ab+bc+ca) = a^2+b^2+c^2 \tag{$*$}$$

Set $a = k \cdot (b+c) + x$ for some positive integer x. Then the above becomes equivalent to:

$$\begin{aligned}
k \cdot ((k \cdot (b+c) + x) \cdot (b+c) + bc) &= (k \cdot (b+c) + x)^2 + b^2 + c^2 \\
k^2 \cdot (b+c)^2 + kx \cdot (b+c) + kbc &= k^2 \cdot (b+c)^2 + 2kx \cdot (b+c) + x^2 + b^2 + c^2 \\
kbc - kx \cdot (b+c) &= x^2 + b^2 + c^2 \\
k \cdot (bc - bx - cx) &= x^2 + b^2 + c^2
\end{aligned}$$

This can be trivially satisfied, if we find positive integers b, c, x with $bc - bx - cx = 1$. Or equivalently:

$$b = \frac{cx+1}{c-x}$$

This shows that for any positive integer x the following integers satisfy $(*)$:

$$\begin{aligned}
c &= x + 1 \\
b &= cx + 1 = x^2 + x + 1 \\
k &= x^2 + b^2 + c^2 = x^4 + 2x^3 + 5x^2 + 4x + 2 \\
a &= k(b+c) + x = x^6 + 4x^5 + 11x^4 + 18x^3 + 20x^2 + 13x + 4
\end{aligned}$$

In particular, any integer of the form $x^4 + 2x^3 + 5x^2 + 4x + 2$ is an element of S. As this grows strictly monotonically with x, we get infinitely many possible positive integer values in S.

Solution 3. to a) Let m be an odd positive number and let $a = F_m$, $b = F_{m+1}$, where F_i denotes the i-th Fibonacci number. Moreover, let

$$c = \frac{(a^2 + ab + b^2)^2 - ab}{a+b} = a^3 + a^2 b + ab^2 + b^3 + ab \cdot \frac{ab-1}{a+b}.$$

In order to prove that c is an integer we will first prove the following identity:

$$F_{k+1}^2 - F_k^2 = F_k F_{k+1} + (-1)^k$$

for all integers $k \geq 1$. Recall that $F_i = \frac{\xi^i - \eta^i}{\xi - \eta}$, where ξ and η are the roots of the quadratic polynomial $t^2 - t - 1$. Then $\xi\eta = -1$. Therefore

$$\begin{aligned}
F_{k+1}^2 - F_k^2 - F_k F_{k+1} &= F_{k+1}(F_{k+1} - F_k) - F_k^2 = F_{k+1} F_{k-1} - F_k^2 \\
&= \frac{\xi^{k+1} - \eta^{k+1}}{\xi - \eta} \cdot \frac{\xi^{k-1} - \eta^{k-1}}{\xi - \eta} - \left(\frac{\xi^k - \eta^k}{\xi - \eta} \right)^2 \\
&= \frac{\xi^{2k} + \eta^{2k} - \xi^{k+1}\eta^{k-1} - \xi^{k-1}\eta^{k+1} - (\xi^{2k} + \eta^{2k} - 2\xi^k \eta^k)}{(\xi - \eta)^2} \\
&= \frac{-\xi^{k-1}\eta^{k-1}(\xi^2 + \eta^2 - 2\xi\eta)}{(\xi - \eta)^2} = \frac{(-1)^k (\xi - \eta)^2}{(\xi - \eta)^2} = (-1)^k.
\end{aligned}$$

Since m is odd we have

$$(b-a)(b+a) = b^2 - a^2 = ab - 1,$$

therefore $a + b \mid ab - 1$ which means that c is indeed an integer.

Now, observe that

$$ab + bc + ca = ab + c(a + b) = (a^2 + ab + b^2)^2.$$

As a consequence,

$$\begin{aligned}(a^2+b^2+c^2)(a+b)^2 &= (a^2+b^2)(a+b)^2 + (c(a+b))^2 \equiv (a^2+b^2)^2 + 2(a^2+b^2)\cdot ab + (ab)^2 \\ &= (a^2+ab+b^2)^2 \equiv 0 \pmod{ab+bc+ca}.\end{aligned}$$

But since $\gcd(a, b) = \gcd(F_m, F_{m+1}) = 1$, then also

$$\gcd(a + b, ab + bc + ca) = \gcd(a + b, ab) = 1.$$

Therefore we obtain that $\frac{a^2+b^2+c^2}{ab+bc+ca}$ is an integer.

To end the proof it suffices to show that $\frac{a^2+b^2+c^2}{ab+bc+ca}$ can be arbitrarily large, depending on the choice of m. But

$$a^2 + b^2 + c^2 \geq c^2 \geq b^6 \quad \text{and}$$

$$ab + bc + ca = (a^2 + ab + b^2)^2 \leq 9b^4,$$

which means that $\frac{a^2+b^2+c^2}{ab+bc+ca} \geq \frac{b^2}{9}$. Since $b = F_{m+1}$ can be arbitrarily large, the conclusion follows.

Solution 1. to b) We will show that $4 \notin S$. It is enough to prove that the equation

$$a^2 + b^2 + c^2 = 4(ab + bc + ca)$$

has no solutions in positive integers. Since squares of integers may be congruent only to 0 or 1 modulo 4, while the right hand side is divisible by 4, we have $a^2 \equiv b^2 \equiv c^2 \equiv 0 \pmod 4$. Therefore $a = 2a_1, b = 2b_1, c = 2c_1$ for some positive integers a_1, b_1, c_1. Then

$$a_1^2 + b_1^2 + c_1^2 = 4(a_1b_1 + b_1c_1 + c_1a_1).$$

By continuing this process we see that a, b, c are divisible by 2^k for every positive integer k, which is a contradiction.

Comment. One can prove in a similar way that $4n \notin S$ for every integer n.

Solution 2. to b) We will prove that $3 \notin S$. We have to show that there are no positive integers $a,\ b,\ c$ satisfying

$$a^2 + b^2 + c^2 = 3(ab + bc + ca).$$

Suppose the contrary and let $a,\ b,\ c$ be a solution to the above that minimizes $a+b+c$. Then at least one of $a,\ b,\ c$ is odd because otherwise $a/2,\ b/2,\ c/2$ is a solution with a smaller sum of variables.

We rewrite the equation in the following form:

$$(a+b)^2 + (b+c)^2 + (c+a)^2 = 8(ab + bc + ca).$$

Since squares of integers may be congruent only to 0 or 1 modulo 4, we see that $(a+b)^2 \equiv (b+c)^2 \equiv (c+a)^2 \pmod 4$. It follows that $a,\ b,\ c$ have the same parity. Since one of $a,\ b,\ c$ is odd, actually all of them are odd. Write $a = 2k+1,\ b = 2l+1,\ c = 2m+1$. Substituting this to the original equation yields

$$4(k^2 + k + l^2 + l + m^2 + m) + 3 = 12(kl + lm + mk + k + l + m) + 9.$$

It follows that $3 \equiv 9 \pmod 4$ which is absurd. Therefore, there are no positive integers $a,\ b,\ c$ satisfying $a^2 + b^2 + c^2 = 3(ab + bc + ca)$.

Comment. One can prove in a similar way that $4n + 3 \notin S$ for every integer n.

Solution 3. to b) Before we start the actual proof, let us give some preparatory statements. First, we may assume without loss of generality that $\gcd(a,b,c) = 1$, as otherwise we can simply divide all of a,b,c by $\gcd(a,b,c)$. Next, we claim that $\gcd(a,b) = 1$. Otherwise, the denominator would be divisible by $\gcd(a,b)$, so the numerator would have to be divisible by $\gcd(a,b)$ as well, which would entail $\gcd(a,b) \mid c$. But this contradicts $\gcd(a,b,c) = 1$.

We now move to the main problem: we claim that $3 \notin S$. In other words, we have to prove that the equation

$$a^2 + b^2 + c^2 = 3ab + 3bc + 3ca$$

has no solutions in positive integers. Regrouping the terms yields

$$c^2 - (3a + 3b)c + a^2 - 3ab + b^2 = 0$$

which we now consider as a quadratic equation in c. For c to be an integer, it is necessary that the discriminant, written below, is a perfect square:

$$\Delta = 9(a+b)^2 - 4(a^2 - 3ab + b^2) = 5(a^2 + 6ab + b^2).$$

This implies that $a^2 + 6ab + b^2 = (a + 3b)^2 - 8b^2$ is divisible by 5. However, $(a + 3b)^2$ may be congruent only to 0, 1, or 4 modulo 5, whereas $8b^2$ may be congruent only to 0, 2, or 3 modulo 5, so their difference can only be divisible by 5 only if $b \equiv 0 \pmod 5$ and $a + 3b \equiv 0 \pmod 5$. This however implies $5 \mid \gcd(a, b) = 1$, which is a contradiction.

Comment. That $4 \notin S$ can be proved in a very similar fashion.

Comment. A (non-exhaustive) computer search found the following integer values smaller than 200 in S:

$$1, 2, 5, 10, 14, 17, 26, 29, 37, 50, 62, 65, 74, 77, 82, 98, 101, 109, 110,$$
$$122, 125, 145, 149, 170, 173, 190, 194, 197$$

Comment. There are several other families of solutions that can be used for part a). Probably the simplest (though maybe not to find...) is

$$\begin{aligned} a &= x + 1 \\ b &= x^2 + 1 \\ c &= x^4 + x^3 + 3x^2 + 2x + 1 \\ n &= x^2 + 1. \end{aligned}$$