Solutions

to Individual Competition Problems

Problem I-1

Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f\big(xf(y)+2y\big) = f(xy) + xf(y) + f\big(f(y)\big)$$

holds for all real numbers x and y.

(proposed by Patrik Bak, Slovakia)

Answer. The functional equation has two solutions, $f(x) \equiv 0$ and $f(x) \equiv 2x$.

Solution. Setting $x = 0$ and $y = 0$ in the functional equation yields $f\left(f(0)\right) = 0$. So there is at least one zero point of f. Let a be any of them. Setting $y = a$ gives us $f(2a) = f(ax) + f(0)$. If $a \neq 0$, then f is a constant function and we know that $f(a) = 0$, so it is a zero function, which is indeed a solution.

It remains to investigate the case where 0 is the only zero point of f, i.e. $f(a) = 0$ if and only if $a = 0$. Furthermore, taking $x = 0$ in the functional equation we obtain

$$f(2y) = f\big(f(y)\big). \tag{1}$$

If we prove an injectivity of f, the previous identity yields $f(y) = 2y$, what is the second solution, as we can easily check.

Now we prove the injectivity of f. Firstly, let us examine the set of the fixed points of f. This set is non-empty because 0 is one of its points. Assume that p is any of the fixed points, i.e. $f(p) = p$. Setting $x = -1$, $y = p$ in the functional equation gives

$$p = f\big(-f(p)+2p\big) = f(-p) - f(p) + f\big(f(p)\big) = f(-p).$$

Now we set $x = 1$, $y = -p$ in the functional equation and we obtain using proved $f(-p) = p$

$$p = f\big(f(-p) - 2p\big) = f(-p) + f(-p) + f\big(f(-p)\big) = 3p.$$

This yields that $p = 0$ is the only fixed point of f.

Secondly, we choose x so that $xf(y) + 2y = xy$, which yields $x = 2y/\big(y - f(y)\big)$. This can be done for each $y \neq 0$, since 0 is the only fixed point of f. This substitution gives us

$$f\big(f(y)\big) = \frac{2yf(y)}{f(y) - y} = \frac{2f^2(y)}{f(y) - y} - 2f(y).$$

In order to finish the proof of injectivity let us assume that non-zero real numbers a, b satisfy $f(a) = f(b)$. We have already proved that $f(a) = f(b) \neq 0$. The previous identity yields

$$\frac{2f^2(a)}{f(a) - a} - 2f(a) = f\big(f(a)\big) = f\big(f(b)\big) = \frac{2f^2(b)}{f(b) - b} - 2f(b)$$

and it follows that $a = b$. The proof of the injectivity is thereby finished.

Remark. We present an alternative way of proving that $f(p) = p$ implies $p = 0$.

Assume that $f(p) = p$. Then $y = p$ in (1) means $f(2p) = p$ and afterwards $y = 2p$ means $f(4p) = p$. Taking $x = 2$ and $y = p$ in the original equation then gives us $p = 0$, which works. Therefore the only fixed point of f is 0.

Solution by Jozef Fülöp awarded by prize of the dean of the FMF, Charles Univerzity, Prague.

We set $x = 1$, $y = 0$ in the original equation and we obtain

$$f\big(f(0)\big) = f(0) + f(0) + f\big(f(0)\big)$$

which implies $f(0) = 0$. This identity yields after setting $x = 0$ in the original equation

$$f(2y) = f\big(f(y)\big). \tag{2}$$

Let us assume that there exists $t_0 \neq 0$ such that $f(t_0) = 0$. For every real number t we obtain by substitution $x = t/t_0$, $y = t_0$ in the original equation

$$f(2t_0) = f(t) + \frac{t}{t_0} \cdot f(t_0) + f\big(f(y)\big) = f(t) + f\big(f(t_0)\big).$$

This equation with (2) gives us $f(t) = 0$ for each real number t, which is one of the solutions, as we can easily check.

Now we can assume that $f(y) \neq 0$ for every real number $y \neq 0$. For each real number t and $y \neq 0$ we obtain by putting $x = 2t/f(y)$ in the original equation

$$f(2t + 2y) = f\left(\frac{2ty}{f(y)}\right) + 2t + f\big(f(y)\big).$$

We prove that the function f is injective. If a, b are numbers from $\mathbb{R} \setminus \{0\}$ such that $f(a) = f(b)\,(\neq 0)$ then the substitutions $t = a$, $y = b$ or $t = b$, $y = a$ in the previous equation gives

$$f(2a+2b) = f\left(\frac{2ab}{f(b)}\right) + 2a + f\big(f(b)\big),$$
$$f(2b+2a) = f\left(\frac{2ba}{f(a)}\right) + 2b + f\big(f(a)\big).$$

This two equations directly yields to $a = b$, which proves the injectivity.

The injectivity of the function f together with (2) lead to $f(y) = 2y$ what is the second solution, as we can check.

Problem I-2

Let $n \geq 3$ be an integer. We say that a vertex A_i $(1 \leq i \leq n)$ of a convex polygon $A_1A_2\ldots A_n$ is *Bohemian* if its reflection with respect to the midpoint of the segment $A_{i-1}A_{i+1}$ (with $A_0 = A_n$ and $A_{n+1} = A_1$) lies inside or on the boundary of the polygon $A_1A_2\ldots A_n$. Determine the smallest possible number of Bohemian vertices a convex n-gon can have (depending on n).

(A convex polygon $A_1A_2\ldots A_n$ has n vertices with all inner angles smaller than $180°$.)

(proposed by Dominik Burek, Poland)

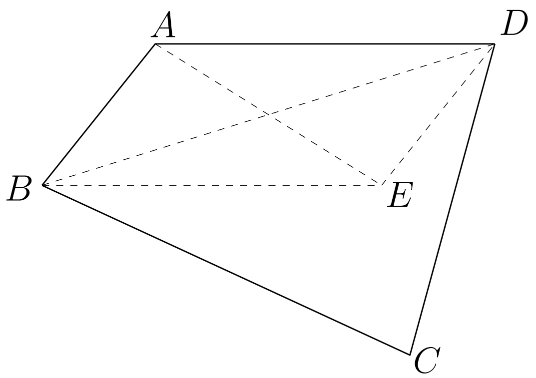
Answer. $n-3$.

In the following we write for short 'reflection of A in P' instead of 'reflection of the vertex A with respect to the midpoint of the segment connecting the two neigbouring vertices of A in the polygon P'.

Solution.

Lemma. *If $ABCD$ is a convex quadrilateral with $\angle BAD + \angle CBA \geq \pi$ and $\angle BAD + \angle ADC \geq \pi$ then A is a Bohemian vertex of $ABCD$.*

Proof. Let E be the reflection of A in $ABCD$. It is clearly seen that E belongs to the halfplanes containing C determined by lines AB and AD. Since $\angle BAD + \angle CBA \geq \pi$ and $\angle BAD + \angle EBA = \pi$, point E belongs to the (closed) halfplane containing points A, D determined by the line BC. Analogously, using the assumption $\angle BAD + \angle ADC$ we infer that E belongs to the closed halfplane containing points A, B determined by the line CD.



Therefore E lies inside or on the boundary of $ABCD$. Thus A is Bohemian.

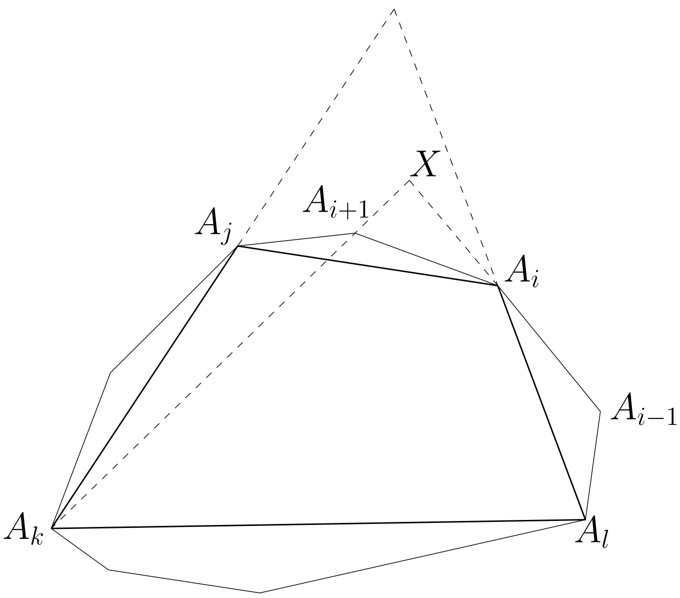
$\square$

Consider a convex n-gon $A_1A_2\ldots A_n$. Choose any four vertices A_i, A_j, A_k, A_l with $i<j<k<l$ as in the picture below. Consider quadrilateral $A_iA_jA_kA_l$. It is clear that one of the points A_i, A_j, A_k, A_l satisfies assumption of the lemma, let's say this point is A_i. We claim that A_i satisfies the assumption of the lemma in quadrilateral $A_{i-1}A_iA_{i+1}A_k$. Observe that the point $X := A_kA_{i+1} \cap A_iA_{i-1}$ lies in the triangle bounded by lines A_kA_j, A_jA_i and A_iA_l. So

$$\angle A_kA_{i+1}A_i + \angle A_{i+1}A_iA_{i-1} = \pi + \angle A_kXA_i \geq \pi.$$

(Note: it may happen that X does not exist. It happens iff $j=i+1$, $l=i-1$ and $A_kA_j \parallel A_lA_i$. In that case $\angle A_kA_{i+1}A_i + \angle A_{i+1}A_iA_{i-1} = \pi$.)

Analogously $\angle A_{i+1}A_iA_{i-1} + \angle A_iA_{i-1}A_k \geq \pi$. Using lemma we conclude that A_i is a Bohemian vertex of quadrilateral $A_{i-1}A_iA_{i+1}A_k$. This implies that A_i is a Bohemian vertex of $A_1A_2\ldots A_n$ since the quadrilateral $A_{i-1}A_iA_{i+1}A_k$ is a subset of the n-gon and the reflexion point is the same.

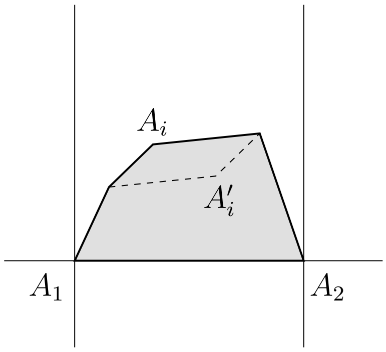


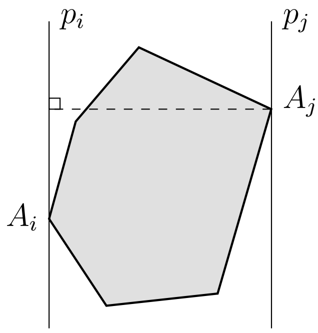
Therefore, amongst any four vertices of a convex n-gon there exists a Bohemian vertex. So, every n-gon has at least $n-3$ Bohemian vertices.

An example of a convex n-gon with exactly $n-3$ Bohemian vertices is the following: take any kite $A_1A_2A_3A_4$ with $A_4A_1 = A_1A_2 < A_2A_3 = A_3A_4$ and place points $A_5, \ldots, A_n$ very close to A_1. Then A_2, A_3, A_4 are not Bohemian vertices of $A_1A_2\ldots A_n$.

Solution 2. We present a sketch of an alternative proof of the fact that the every n-gon has *at least* $n-3$ Bohemian vertices.

Observation 1. Let us place the polygon into a coordinate system in such a way that $A_1 = [0,0]$, $A_2 = [a,0]$, $a>0$ and the second coordinates of all the remaining vertices are positive. If all the remaining vertices A_2, $\ldots$, A_n have their first coordinates between 0 and a (see picture below), it is easy to see that the only vertices that could be non-Bohemian are A_1, A_2, and the point with the strictly largest second coordinate (if such a vertex exists). So, in this case, there exist at least $n-3$ Bohemian vertices.





Observation 2. An affine transformation does not change anything, so the statement is proved for all polygons that lie between two parallel lines that go through two adjacent vertices, i.e., whenever there are two adjacent vertices with sum of their angles at most 180°.

Consider now any polygon $P = A_1A_2, \dots A_n$.

Observation 3. If there are two (non-adjacent) vertices A_i, A_j and two parallel lines p_i, p_j with $A_i \in p_i$, $A_j \in p_j$ such that the whole polygon lies between p_i and p_j, then the diagonal A_iA_j splits P into two polygons of the type considered in Observation 2. By Observations 1 and 2, these two polygons have at most 4 non-Bohemian points together, A_i, A_j, and two more.

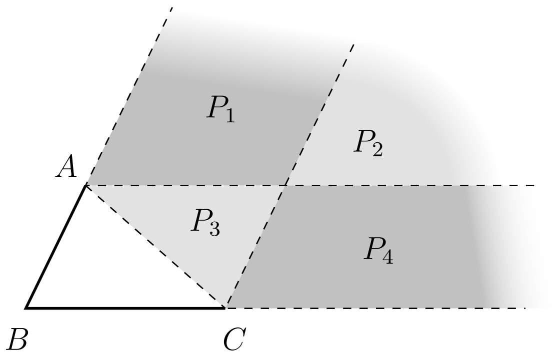
Observation 4. For any vertex A_i there exist a vertex A_j and two parallel lines p_i and p_j with $A_i \in p_i$, $A_j \in p_j$ such that the whole polygon lies between them. In fact, take a line p_i such that $p_i \cap P = \{A_i\}$, then A_j is the vertex with maximal distance from p_i, if there are two such vertices, change the direction of p_i slightly to obtain a unique A_j.

Let $n = 4$. Then there exist two adjacent vertices with sum of their angles not larger than 180°, so, by Observation 2, any quadrilateral has at most 3 non-Bohemian vertices.

Let $n \geq 5$. By Observations 3 and 4, there exist at most 4 non-Bohemian vertices. So, at least one vertex is Bohemian, denote it by A_i. Then, by Observations 3 and 4, all the non-Bohemian vertices are contained in the quadruple A_i, A_j and some other two vertices. Since A_i is Bohemian, there are at most three non-Bohemian vertices and the proof is complete.

Solution 3. We prove by induction that every n-gon has at least $n-3$ Bohemian vertices.

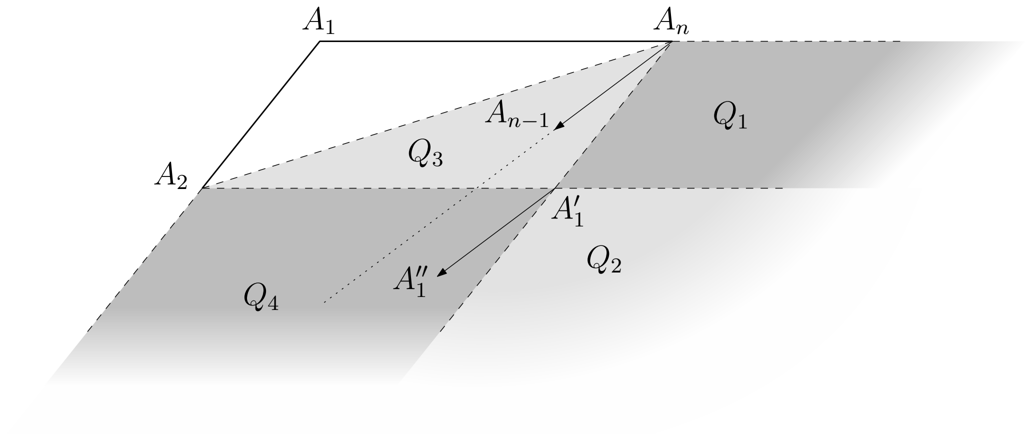
Step 1, $n = 4$. We show that every quadrilateral $ABCD$ has at least one Bohemian vertex. We consider a triangle ABC. Then D has to be in one of the areas P_1, P_2, P_3, P_4 (see picture below), otherwise, $ABCD$ would not be a convex quadrilateral. If D were in P_2, then B would be Bohemian. If D were in P_3, it would be Bohemian. If D were in P_1, then A would be Bohemian and similarly if D were in P_4, then C would be Bohemian (the last two cases are not immediate but easy to prove).



Induction step. Consider an n-gon $P = A_1A_2 \dots A_n$ with $n \geq 5$. Let P' be an $(n-1)$-gon obtained from P by omitting one vertex different from A_1. Let A_1' be the reflexion of A_1 in P and A_1'' the reflexion of A_1 in P'. We show the following statement

$$\text{if } A_1 \text{ is non-Bohemian in } P, \text{ then it is non-Bohemian also in } P'. \tag{S}$$

This statment is obvious if the omitted vertex is not adjacent to A_1 (since in this case $A_1' = A_1''$ and $P' \subset P$). So, let the omitted vertex be A_n (the other neighbour A_2 can be done in the same way) and let us assume for contradiction that $A_1' \notin P$ and $A_1'' \in P'$. Let us observe that vectors A_nA_{n-1} and $A_1'A_1''$ are equal. Let us discuss the possible position of A_{n-1}. If $A_{n-1} \in Q_3 \cup Q_4$ (as in the picture below) then A_1'' lies below the line A_nA_{n-1} while the whole polygon P lies above this line, contradiction with $A_1'' \in P' \subset P$. If $A_{n-1} \in Q_2$, then $A_1' \in \triangle A_2A_nA_{n-1} \subset P$, contradiction. If $A_{n-1} \in Q_1$, then the new reflexion $A_1'' \in Q_2$ and $A_1' \in \triangle A_2A_nA_1''$ and $\triangle A_2A_nA_1'' \subset P'$ since $A_1'' \in P'$. Therefore $A_1' \in P' \subset P$, contradiction. Statement (S) is proved.



Since (by the induction hypothesis) there are at most 3 non-Bohemian vertices in P', there are at most 4 non-Bohemian vertices in P (the three and the omitted one). Since $n \geq 5$ there is at least one Bohemian vertex in P. Assume now that P' is obtained form P by omitting a Bohemian vertex. Since there are at most 3 non-Bohemian vertices in P', there are at most 3 non-Bohemian vertices in P and the proof is complete.

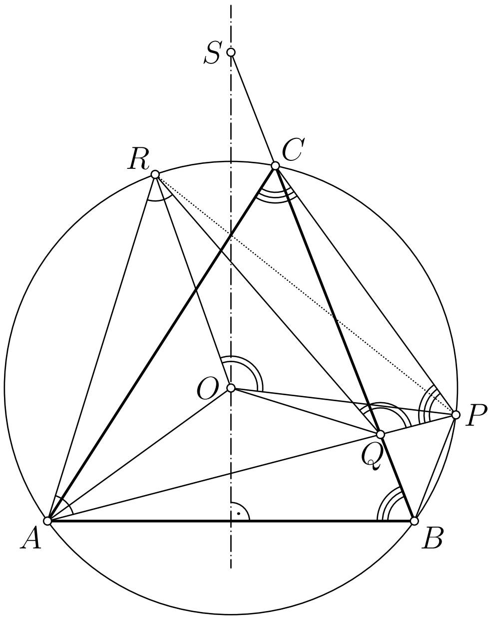
Problem I-3

Let ABC be an acute-angled triangle with $AC > BC$ and circumcircle ω. Suppose that P is a point on ω such that $AP = AC$ and that P is an interior point of the shorter arc BC of ω.

Let Q be the point of intersection of the lines AP and BC. Furthermore, suppose that R is a point on ω such that $QA = QR$ and that R is an interior point of the shorter arc AC of ω. Finally, let S be the point of intersection of the line BC with the perpendicular bisector of the side AB. Prove that the points P, Q, R, and S are concyclic.

(proposed by Patrik Bak, Slovakia)

Solution. Let ud denote O the center of the circle ω and $\varphi = \angle PAR$. Since the triangle QAR is isosceles, we have $\angle ARQ = \varphi$ and $\angle PQR = 2\varphi$. The central angle theorem (applying to the chord PR) also yields $\angle POR = 2\varphi$. Thus the points P, Q, O and R are concyclic.



Further, let us denote $\beta = \angle ABC$. Since $AP = AC$, we have $\angle ACP = \angle APC = \beta$ and thus (by the central angle theorem) $\angle AOP = 2\beta$, which gives

$$\angle PAO = \angle APO = 90^\circ - \beta = \angle OPQ.$$

Since $\angle ABS = \beta$, we furthermore have $\angle OSB = \angle OSQ = 90^\circ - \beta$, which concludes $\angle OPQ = \angle OSB$ and therefore also the points P, Q, O and S are concyclic.

From both paragraphs above it immediately follows the requested claim, i.e. the points P, Q, R, S are concyclic, and the proof is done.

Problem I-4

Determine the smallest positive integer n for which the following statement holds true: From any n consecutive integers one can select a non-empty set of consecutive integers such that their sum is divisible by 2019.

(proposed by Kartal Nagy, Hungary)

Answer. $n = 340$.

Solution The prime factorization of 2019 is $3 \cdot 673$. Let $p = 673$.

For each integer k, color the three numbers $kp-1, kp, kp+1$ red, and and the six numbers $kp+\frac{p-5}{2}, kp+\frac{p-3}{2}, kp+\frac{p-1}{2}, kp+\frac{p+1}{2}, kp+\frac{p+3}{2}, kp+\frac{p+5}{2}$ blue. Now the integers are colored periodically. In a period of length $p = 673$, there are 3 red integers, then 332 uncolored integers, then 6 blue integers and finally 332 uncolored integers.

The sum of the integers in a red interval is $3kp = 2019 \cdot k$, and the sum of the integers in a blue interval is $6(kp + \frac{p}{2}) = 2019 \cdot (2k+1)$. So if there is a colored interval (we mean a maximal one throughout) in the given n consecutive integers, one can choose it. It is easy to see, that among any $340 = 332 + (6-1) + (3-1) + 1$ consecutive integers, there must be a colored interval. Thus the smallest n (that we look for) satisfies $n \leq 340$.

Now we will show that it is not possible to choose consecutive integers in the desired way from the set $A = \{335, 336, \ldots, 673\}$. ($|A| = 339$ and thus $n \geq 340$.) Assume that there exists $\{a, a+1, \ldots b\} \subseteq A$ such that

$$2019 \mid a + (a+1) + \cdots + b = \frac{(b-a+1)(a+b)}{2}.$$

That means either $673 \mid b-a+1$, or $673 \mid a+b$. Since

$$0 < 1 \leq b-a+1 \leq 339 < 673,$$

673 must divide $a+b$. Taking into account that

$$671 = 335 + 336 \leq a+b \leq 673 + 673 = 2 \cdot 673,$$

we conclude that $a+b$ must be 673 or $2 \cdot 673$. It means either $a = 335$ and $b = 338$, or $a = 336$ and $b = 337$, or $a = b = 673$. But $2019 \nmid 335 + 336 + 337 + 338 = 1346$, $2019 \nmid 336 + 337 = 673$ and $2019 \nmid 673$, a contradiction.

Comment. The same proof works for every odd number $m = p \cdot q$, where p is a 'big' prime divisor of m. We need that $p > \sqrt{3m}$. Then the answer is $n = \frac{p+3q}{2} - 1$.

Solutions

to Team Competition Problems

Problem T-1

Determine the smallest and the greatest possible values of the expression

$$\left(\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\right)\left(\frac{a^2}{a^2+1}+\frac{b^2}{b^2+1}+\frac{c^2}{c^2+1}\right)$$

provided a, b, and c are non-negative real numbers satisfying $ab+bc+ca=1$.

(proposed by Walther Janous, Austria)

Answer. The smallest value is $\frac{27}{16}$ and the greatest is 2.

Solution. Let us denote

$$x=\frac{a^2}{a^2+1}+\frac{b^2}{b^2+1}+\frac{c^2}{c^2+1}, \qquad y=\frac{2abc}{(a+b)(b+c)(c+a)}$$

to simplify notation.

Denominators in x can be manipulated using $ab+bc+ca=1$ as

$$a^2+1=a^2+ab+bc+ca=(a+b)(a+c)$$

and similarly for b^2+1 and c^2+1. This yields a relation between x and y

$$x=\frac{a^2(b+c)+b^2(c+a)+c^2(a+b)}{(a+b)(b+c)(c+a)}=1-\frac{2abc}{(a+b)(b+c)(c+a)}=1-y.$$

Using

$$\frac{1}{a^2+1}=1-\frac{a^2}{a^2+1}$$

and similar relations for b and c we have

$$\left(\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{b^2+1}\right)\left(\frac{a^2}{a^2+1}+\frac{b^2}{b^2+1}+\frac{c^2}{c^2+1}\right)=(3-x)x.$$

Since $(3-x)x=(2+y)(1-y)=2-y-y^2$, we want to estimate y. Obviously $y\geq 0$ with equality e.g. for $(a,b,c)=(0,1,1)$. On the other hand $y\leq \frac{1}{4}$ as can be seen after multiplying three AM-GM inequalities

$$a+b\geq 2\sqrt{ab},\qquad b+c\geq 2\sqrt{bc},\qquad c+a\geq 2\sqrt{ca}.$$

The equality is reached for $a=b=c=\sqrt{\frac{1}{3}}$.

Finally we compute

$$\frac{27}{16}=2-\frac{1}{4}-\frac{1}{16}\leq 2-y-y^2\leq 2.$$

Similar solution. Let us denote

$$x=\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}.$$

Since

$$\frac{a^2}{a^2+1}=1-\frac{1}{a^2+1}$$

holds we have

$$\left(\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\right)\left(\frac{a^2}{a^2+1}+\frac{b^2}{b^2+1}+\frac{c^2}{c^2+1}\right)=x(3-x)=\frac{9}{4}-\left(x-\frac{3}{2}\right)^2.$$

To obtain the extrema it is sufficient to find the bounds of x.

The sum $a+b+c$ is non-negative because $a,b,c\geq 0$. If $a+b+c=0$ then $a=b=c=0$ what is in contradiction with $ab+bc+ca=1$, so $a+b+c>0$. Using $ab+bc+ca=1$ we obtain

$$\frac{1}{a^2+1}=\frac{1}{a^2+ab+bc+ca}=\frac{1}{(a+b)(a+c)}$$

and similarly for $1/(b^2+1)$ and $1/(c^2+1)$. This yields

$$x=\frac{2(a+b+c)}{(a+b)(b+c)(c+a)}=\frac{2(a+b+c)}{(a+b+c)(ab+bc+ca)-abc}=\frac{2}{1-\frac{abc}{a+b+c}}$$

and the problem is reduced to find bounds of $abc/(a+b+c)$. We have obviously

$$0\leq\frac{abc}{a+b+c}.$$

The equality holds there if one of a, b, c is equal 0 and the product of remaining two is 1. On the other hand using AM-GM inequality we have

$$a+b+c=(a+b+c)(ab+bc+ca)\geq 3\,(abc)^{\frac{1}{3}}\cdot 3\,(abc)^{\frac{2}{3}}=9abc$$

with equality in the case $a = b = c = \frac{\sqrt{3}}{3}$. So

$$\frac{abc}{a+b+c} \leq \frac{1}{9}.$$

These inequalities give us the bounds for x

$$2 \leq x = \frac{2}{1 - \frac{abc}{a+b+c}} \leq \frac{9}{4}.$$

This follows

$$\frac{27}{16} \leq \frac{9}{4} - \left(x - \frac{3}{2}\right)^2 \leq 2.$$

The lower bound arises if one of a, b, c is zero and product of the others is 1, the upper bound arises if $a = b = c = \frac{\sqrt{3}}{3}$.

Problem T-2

Let α be a real number. Determine all polynomials P with real coefficients such that

$$P(2x + \alpha) \leq (x^{20} + x^{19})\, P(x)$$

holds for all real numbers x.

(proposed by Walther Janous, Austria)

Answer. For all α the only satisfying polynomial is $P(x) \equiv 0$.

Solution. Zero polynomial obviously satisfies the problem. Further, let us suppose that polynomial P is non-zero. Let n be its degree and $a_n \neq 0$ be its coefficient at x^n. Polynomial $(x^{20} + x^{19})P(x) - P(2x + \alpha)$ has degree $n + 20$, coefficient a_n at x^{n+20} and it is non-negative for all real numbers x. It follows that $n + 20$ (and n too) is an even number and $a_n > 0$.

For $x = -1$ and $x = 0$ we obtain

$$P(-2 + \alpha) \leq 0 \quad \text{and} \quad P(\alpha) \leq 0.$$

So P has real roots. Let m be its minimal real root and M the maximal real root. Since $a_n > 0$ the values $P(x)$ are positive outside the interval $\langle m, M\rangle$. It yields $\{-2 + \alpha, \alpha\} \subset \langle m, M\rangle$, the interval $\langle m, M\rangle$ is so proper (non-degenerate) and it has the length at least 2.

For $x = m$ we have

$$P(2m + \alpha) \leq 0.$$

This implies $m \leq 2m + \alpha$ and therefore $-\alpha \leq m$. Analogously for $x = M$ we obtain

$$P(2M + \alpha) \leq 0.$$

This yields $2M + \alpha \leq M$ and $M \leq -\alpha$. It follows altogether $m = M = -\alpha$, which contradicts the fact that $\langle m, M \rangle$ is the proper interval. This finally proves that non-zero polynomial P satisfying the problem does not exist.

Problem T-3

There are n boys and n girls in a school class, where n is a positive integer. The heights of all the children in this class are distinct. Every girl determines the number of boys that are taller than her, subtracts the number of girls that are taller than her, and writes the result on a piece of paper. Every boy determines the number of girls that are shorter than him, subtracts the number of boys that are shorter than him, and writes the result on a piece of paper. Prove that the numbers written down by the girls are the same as the numbers written down by the boys (up to a permutation).

(proposed by Stephan Wagner, Austria)

Solution. We prove the statement by induction. The case $n = 1$ is easy (either both children write down 0, or both write down 1). For the induction step, suppose that the children are standing in a row in order of height (the tallest first), and consider a boy and a girl standing next to each other (such a pair must clearly always exist). If k boys and ℓ girls are taller than these two, then either both write down $k - \ell = (n - \ell - 1) - (n - k - 1)$ (if the girl is taller), or both write down $(k+1) - \ell = (n - \ell) - (n - k - 1)$ (if the boy is taller).

If we remove these two children from the class, the numbers of all the other children would remain the same (the boy and the girl cancel in the other children's calculations). Thus we are done by the induction hypothesis.

Remark. This solution can be modified in many ways. For example, instead of removing the two children, we can let them "switch heights". This can be repeated until we reach the situation that all boys are taller than all girls (or vice versa), in which case the numbers are easy to determine.

Problem T-4

Prove that every integer from 1 to 2019 can be represented as an arithmetic expression consisting of up to 17 symbols 2 and an arbitrary number of additions, subtractions, multiplications, divisions and brackets. The 2's may not be used for any other operation, for example to form multi-digit numbers (such as 222) or powers (such as 2^2).

Valid examples:

$$\left((2 \times 2 + 2) \times 2 - \frac{2}{2}\right) \times 2 = 22, \quad (2 \times 2 \times 2 - 2) \times \left(2 \times 2 + \frac{2+2+2}{2}\right) = 42.$$

(proposed by Stephan Wagner, Austria)

Solution 1. We will first prove by induction that every even number less than 2^n can be written with at most $\frac{3}{2}n-1$ 2's. This is certainly true for $n=2$ and $n=3$ with $2=2$, $4=2+2$ and $6=2+2+2$.

Let $k \geq 8$ be an even number $< 2^n$. If it is divisible by 4, it can be written as $2(\frac{k}{2})$ which needs at most $1+\frac{3}{2}(n-1)-1 < \frac{3}{2}n-1$ by induction. If $k \equiv 2 \pmod 4$, then $k = 2+2\cdot 2k'$ where $k' < 2^{n-2}$. If k' is an even number, then we obtain k using at most $1+2+\frac{3}{2}(n-2)-1 = \frac{3}{2}n-1$ 2's, by induction. If k' is odd, then $k'+1$ is even and we have $k = 2\cdot 2(k'+1)-2$. If $k'+1 < 2^{n-2}$ then we can use induction again to get k with at most $1+2+\frac{3}{2}(n-2)-1 = \frac{3}{2}n-1$ 2's. If $k'+1 = 2^{n-2}$, we obtain $k = 2\cdot 2\cdot 2^{n-2}-2$ using $n+1$ 2's which is less or equal to $\frac{3}{2}n-1$ since $n \geq 4$. This finishes the proof for even numbers.

Obviously, any odd number can be obtained from an even number by adding $\frac{2}{2}$, so any odd number less than 2^n can be obtained by at most $\frac{3}{2}n-1+2 = \frac{3}{2}n+1$ 2's, which for $n=11$ yields 17 2's.

Solution 2. It is enough to show that all multiples of 4 can be written using at most 15 2's (since numbers not divisible by 4 can be written as $N+2$ or $N \pm \frac{2}{2}$ where N is divisible by 4). So, let N be divisible by 4 and let its binary representation be $N = 2^{a_1}+\cdots+2^{a_k}$ with $a_1 > a_2 > \cdots > a_k > 1$. Since $N < 2019$ we have $a_1 \leq 10$. Observe that k is the number of 1's in the binary representation of N. We distinguish two cases:

1st case: Let $k \leq 6$, i.e. there are at most six 1's in the binary representation of N. Then we can write

$$N = 2^{a_k-1}(2+2^{a_{k-1}-a_k}(2+2^{a_{k-2}-a_{k-1}}(2+\ldots(2+2^{a_1-a_2+1})))).$$

If we rewrite all powers using multiplication, we obtain an expression with $a_k-1+a_{k-1}-a_k+\cdots+a_1-a_2+1 = a_1$ 2's comming from powers and one additional 2 added in each bracket. Since there are $k-1$ brackets, we need $a_1+k-1 \leq 10+5 = 15$ 2's to represent N.

2nd case: Let $k \geq 7$. Then we can write $N = 2^{11}-1-2^{b_1}-2^{b_2}-\cdots-2^{b_l}$, where $b_1 > b_2 > \cdots > b_l$ are the positions of zeros in the binary representation of N. Since N is divisible by 4, we have $b_l = 0$, $b_{l-1} = 1$. Similarly to the first case, we have

$$N = 2^{11}-2^{b_1}-\cdots-2^{b_{l-2}}-4 = 2(2^{b_{l-2}-2}(2^{b_{l-3}-b_{l-2}}(\ldots(2^{11-b_1+1}-2)\ldots)-2)-2).$$

If we expand powers into multiplications, the number of multiplicating 2's is $1+(b_{l-2}-2)+(b_{l-3}-b_{l-2})+\cdots+11-b_1+1 = 11$ and the number of 2's after minus signs in exactly $l-1$, i.e. at most $10+l$ 2's in total. Since $k \geq 7$, there are at most four zeros in the binary representation of N, i.e. $l \leq 4$ and $10+l \leq 15$, which completes the proof.

Problem T-5

Let ABC be an acute-angled triangle such that $AB < AC$. Let D be the point of intersection of the perpendicular bisector of the side BC with the side AC. Let P be a point on the shorter arc AC of the circumcircle of the triangle ABC such that $DP \parallel BC$. Finally, let M be the midpoint of the side AB. Prove that $\angle APD = \angle MPB$.

(proposed by Dominik Burek, Poland)

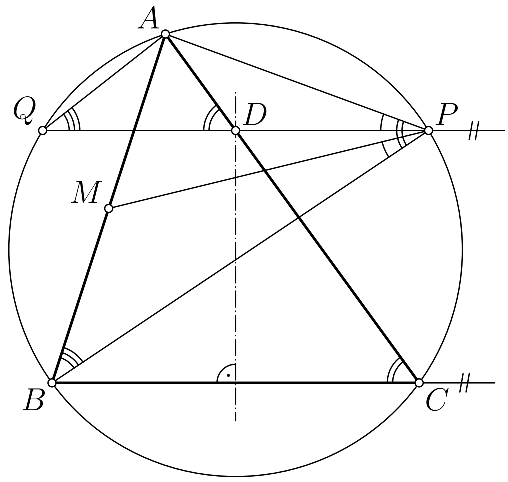
Solution. Let the line DP intersects the cirmumcircle of the triangle ABC again at a point Q. We can see that

$$\angle ADQ = \angle ACB = \angle APB \qquad \text{and} \qquad \angle AQD = \angle AQP = \angle ABP.$$

This implies that the triangles AQD and ABP are similar and therefore the equality

$$\frac{AQ}{QD} = \frac{AB}{BP}$$

holds.



Since $QP = 2\,QD$ and $AB = 2\,MB$ we immediately obtain

$$\frac{AQ}{QP} = \frac{MB}{BP}$$

Since further $\angle AQP = \angle MBP$, the triangles AQP and MBP are also similar. This yields $\angle APD = \angle MPB$ and the proof is finished.

Remark. We can also prove that the line DP is the symmedian in vertex P of the triangle ABP. For this purpose we can consider the point X of intersection of tangents to the circumcircle of the triangle ABP at the vertices A and B, and then we can easily prove that the points P, D and X are collinear.

Problem T-6

Let ABC be a right-angled triangle with its right angle at B and circumcircle c. Denote by D the midpoint of the shorter arc AB of c. Let P be the point on the side AB such that $CP = CD$ and let X and Y be two distinct points on c satisfying $AX = AY = PD$. Prove that the points X, Y and P are collinear.

(proposed by Dominik Burek, Poland)

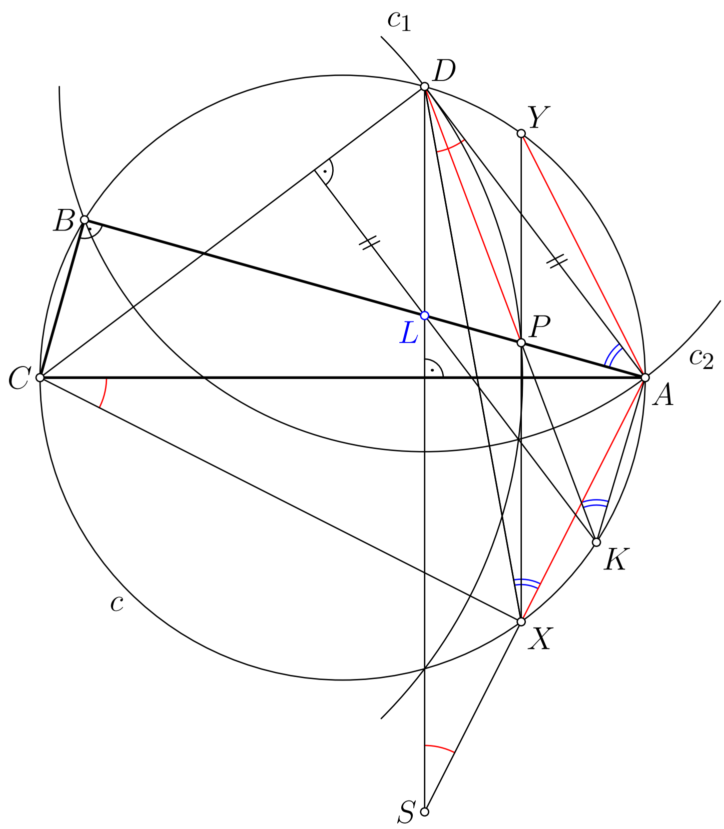
Solution. It will be enough to prove that $PX \perp AC$. Without loss of generality assume that X lies in another half-plane with regard to the line AC than the point B. Let us denote $K \neq D$ the point of intersection of the line DP and the circle c. Further, let S be the point on the ray AX with $AS = DK$. Since D is the midpoint of the arc AB, we have

$$\angle BAD = \angle PAD = \angle AKD = \angle AXD$$

and thus the triangles DAP and DKA are (by the AA-theorem) similar. Therefore it holds $AD^2 = DP \cdot DK = AX \cdot AS$. This implies that the triangles AXD and ADS are also similar. Hence

$$\angle DSA = \angle ADX = \angle ACX = 90^\circ - \angle XAC = 90^\circ - \angle SAC.$$

Thus $SD \perp AC$.



Let c_1 be the circle with center C and radius $CD = CP$. Similarly, let c_2 be the circle with center D and radius $DA = DB$. Finally, let L be a point of intersection of the lines SD and AB. Then DL (i.e. DS) is the radical axis of c_1 and c, because D lies on both circles (c_1 and c) and DS is perpedicular to the line passing through the centres of c_1 and c.

Moreover, AB is the radical axis of c_2 and c since both circles pass through A and B. Hence the point L is the radical center of the cicles c, c_1 and c_2. For powers of the point K with respect to the circles c_1 and c_2 we therefore have

$$\operatorname{Pow}(K, c_2) = DK^2 - DA^2 = DK^2 - DP \cdot DK = KP \cdot KD = \operatorname{Pow}(K, c_1).$$

Thus KL is the radical axis of c_1 and c_2. In particular, KL is perpendicular to the line passing through centres of these two circles. i.e. $KL \perp CD$. Since $AD \perp CD$, we get $KL \parallel AD$. Similarity of the triangles KLP and DAP (the quadrilateral $KLDA$ is a trapezoid with the intersection point P of its diagonals) further yields

$$\frac{AP}{AL} = \frac{DP}{DK} = \frac{AX}{AS},$$

thus $PX \parallel DS$, so $PX \perp AC$, and the proof is done.

Remark. To prove $PX \perp AC$ there exist also many computational solutions using Pythagorean theorem, analytical geometry or complex numbers.

Problem T-7

Let a, b and c be positive integers satisfying $a < b < c < a + b$. Prove that $c(a-1)+b$ does not divide $c(b-1)+a$.

(proposed by Dominik Burek, Poland)

Solution 1 Put $A = c(a-1)+b$, $B = c(b-1)+a$ and suppose that A is a divisor of B. Then A is also a divisor of the number $C = bA - aB$. Since

$$C = b(c(a-1)+b) - a(c(b-1)+a) = (b-a)(a+b-c) > 0,$$

it follows from $c > b-a > 0$ and $a-1 \geq a+b-c > 0$ that

$$A = c(a-1)+b > c(a-1) > (b-a)(a+b-c) = C.$$

Thus $A > C > 0$, which implies that A does not divide C, a contradiction.

Solution 2 It suffices to verify that

$$\frac{b-1}{a} < \frac{c(b-1)+a}{c(a-1)+b} < \frac{b}{a},$$

because no integer lies between the two fractions $\frac{b-1}{a}$ and $\frac{b}{a}$. Routine algebraic manipulations show that the left-hand inequality is equivalent to

$$c > b - \frac{a^2}{b-1}, \quad \text{where} \quad \frac{a^2}{b-1} > 0,$$

while the right-hand inequality is equivalent to $c < a+b$. The proof is complete.

Solution 3 Put $A = c(a-1)+b$, $B = c(b-1)+a$ and suppose that A divides B. We will prove by induction that for any positive integer n, both inequalities $b \geq na$ and $B \geq nA$ hold true. It is clear that no such a and b exist, since $a \geq 1$ and thus $b < na$ for some n.

For $n = 1$, we have $b > a$ by the conditions of the problem. Besides, since $A \mid B$ and A, B are clearly positive, we have $B \geq A$ as well.

Let $n \geq 1$ be now an integer such that $b \geq na$ and $B \geq nA$. Our goal is to prove that $b \geq (n+1)a$ and $B \geq (n+1)A$ as well. Firstly we verify that $B > nA$. If $b > na$, then

$$B - nA = c(b - 1 - an) + cn + a - nb \geq cn + a - nb = n(c - b) + a > a > 0$$

and we are done. On the other hand, if $b = na$, then $nb - a = (n-1)(a+b)$ and hence the nonnegative number $B - nA$ can be written as

$$B - nA = (n-1)c - (nb - a) = (n-1)c - (n-1)(a+b) = (n-1)(c - a - b),$$

which means that $n = 1$ (because of $c < a + b$), which contradicts to $b = na$. So $B > nA$ is proven. Since $A \mid (B - nA)$, we have $B - nA \geq A$, i.e. $B \geq (n+1)A$. To finish the second induction step, it remains to prove that $b \geq (n+1)a$.

The proved inequality $B \geq (n+1)A$ means that

$$c\Big(b - (n+1)a + n\Big) \geq (n+1)b - a.$$

Since $b \geq na$ implies that $a \leq \frac{b}{n}$ and hence $\frac{n+1}{n}b \geq a + b > c$, we can conclude the following:

$$(n+1)b - a \geq \Big((n+1) - \tfrac{1}{n}\Big)\, b = \tfrac{n(n+1)-1}{n+1} \cdot \tfrac{n+1}{n} b > \tfrac{n(n+1)-1}{n+1} c = \Big(n - \tfrac{1}{n+1}\Big)\, c.$$

Comparing this with the preceding inequality, we get

$$b - (n+1)a + n > n - \tfrac{1}{n+1}, \quad \text{or} \quad b - (n+1)a > -\tfrac{1}{n+1} > -1,$$

hence the integer $b - (n+1)a$ is nonnegative, as we wished to prove.

Problem T-8

Let N be a positive integer such that the sum of the squares of all positive divisors of N is equal to the product $N(N+3)$. Prove that there exist two indices i and j such that $N = F_i \cdot F_j$, where $(F_n)_{n=1}^{\infty}$ is the Fibonacci sequence defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$.

(proposed by Alain Rossier, Switzerland)

Solution Denote by $1 = d_1 < d_2 < \cdots < d_k = N$ all positive divisors of the given positive integer N that satisfies

$$d_1^2 + d_2^2 + \cdots + d_k^2 = N(N+3), \quad \text{i.e.} \quad d_2^2 + d_3^2 + \cdots + d_{k-1}^2 = 3N - 1.$$

Now $3N - 1 > 0$ implies that $k \geq 3$. However, if $k = 3$, then $N = p^2$ with some prime $p = d_2$ satisfying $p^2 = 3p^2 - 1$, which is impossible. Thus $k \geq 4$.

For each $i = 2, 3, \ldots, k-1$, we have $d_i d_{k+1-i} = N$ and hence $d_i^2 + d_{k+1-i}^2 \geq 2N$ by the AM-GM inequality. Consequently,

$$3N - 1 = \sum_{i=2}^{k-1} d_i^2 = \frac{1}{2}\sum_{i=2}^{k-1}\Big(d_i^2 + d_{k+1-i}^2\Big) \geq \frac{1}{2}(k-2) \cdot 2N = (k-2)N.$$

However, $3N-1 \geq (k-2)N$ means that $k \leq 5 - \frac{1}{N} < 5$, which together with the previous fact that $k \geq 4$ leads to the equality $k = 4$. Thus either $N = p^3$ with p being a prime and our equation becomes $p^2 + p^4 = 3p^3 - 1$, or N has a factorization $N = pq$, with some primes $p > q$ and our equation becomes $p^2 + q^2 = 3pq - 1$. As the first case cannot have any solutions, we conclude that the latter must be true.

Notice that the equation $p^2 + q^2 = 3pq - 1$ has 'prime' solutions $(p, q) = (5, 2) = (F_5, F_3)$ and $(p, q) = (13, 5) = (F_7, F_5)$. This encourages us to prove a more general fact: Any solution (a, b) of the equation $a^2+b^2 = 3ab-1$ with positive integers $a > b$ is of the form $(a, b) = (F_{2i+1}, F_{2i-1})$, for some $i \geq 1$. After proving it, we get the desired representation $N = pq = F_{2i+1}F_{2i-1}$.

Assume to the contrary that there exist integers $a > b > 0$ such that $a^2 + b^2 = 3ab - 1$ but no i such that $a = F_{2i+1}$ and $b = F_{2i-1}$. Among all such pairs (a, b), take the one with b minimal. By Vieta's formulas, the equation $a^2 + b^2 = 3ab - 1$ remains to hold if we replace a by the number a' that satisfies $a + a' = 3b$ and $aa' = b^2 + 1$. In view of the symmetry, the solutions of the equation are then not only pairs (a, b) and (a', b), but (b, a') as well.

Note that the number $a' = 3b - a$ is an integer which is positive, because of $a > 0$ and $aa' = b^2 + 1 > 0$. Moreover, $a \geq b + 1$ implies that

$$a' = \frac{b^2+1}{a} \leq \frac{b^2+1}{b+1} = b - \frac{b-1}{b+1} \leq b.$$

Thus $a' < b$, excepting the case when $a = b + 1$ and $b = 1$, i.e. $(a, b) = (2, 1) = (F_3, F_1)$, but this is in contradiction to our choice of (a, b). The established inequalities $0 < a' < b$ show that the new pair (b, a') satisfies all the conditions above. By the minimality there must exist i such that $(b, a') = (F_{2i+1}, F_{2i-1})$. As a consequence, we get

$$a = 3b - a' = 3F_{2i+1} - F_{2i-1}.$$

However, we easily verify that $3F_{2i+1} - F_{2i-1} = F_{2i+3}$ for any $i \geq 1$. After doing it, we conclude that $(a, b) = (F_{2i+3}, F_{2i+1})$, contrary to the choice of (a, b).

To complete our solution, it remains to prove the identity $3F_{2i+1} - F_{2i-1} = F_{2i+3}$. Putting $u = F_{2i-1}$ and $v = F_{2i}$, we successively get $F_{2i+1} = u + v$, $F_{2i+2} = u + 2v$, $F_{2i+3} = 2u + 3v$, so

$$3F_{2i+1} - F_{2i-1} = 3(u + v) - u = 2u + 3v = F_{2i+3}.$$