MEMO Online

14th Middle European Mathematical Olympiad 2020

Contest Problems with Solutions

The Problem Selection Committee

The Problem Selection Committee

	Algebra	**Combinatorics**	**Geometry**	**Number Theory**
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gratefully received

63 problem proposals submitted by 5 countries:

Austria — Croatia — Czech Republic — Poland — Slovakia

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Contents

I-1

Let $\mathbb{N}$ be the set of positive integers. Determine all positive integers k for which there exist functions $f\colon \mathbb{N} \to \mathbb{N}$ and $g\colon \mathbb{N} \to \mathbb{N}$ such that g assumes infinitely many values and such that

$$f^{g(n)}(n) = f(n) + k$$

holds for every positive integer n.

(*Remark.* Here, f^i denotes the function f applied i times, i.e., $f^i(j) = \underbrace{f(f(\ldots f(f}_{i\text{ times}}(j))\ldots))$.)

Answer. Such functions exist if and only if $k \geq 2$.

Solution 1. Suppose that $k = 1$ and that f and g satisfy the desired conditions.

Claim. *There exist no positive integers m and n with $f^m(n) = n$.*

Proof. Suppose that $f^m(n) = n$ for some positive integers m and n. Consider the orbit of n, i.e. the set $A = \{f(n), f^2(n), \ldots, f^{m-1}(n), f^m(n)\}$. Clearly, $f^r(n) \in A$ for all integers $r \geq 0$. Let $t = f^s(n) = \max A$ and denote $u = f^{s+m-1}(n)$ so that $f(u) = t$. Then $t + 1 = f(u) + 1 = f^{g(u)}(u) = f^{g(u)+s+m-1}(n) \in A$, contradicting $t = \max A$. □

Claim. $f(n) \geq n + 1$ *for all positive integers n.*

Proof. Suppose there exists a positive integer n with $f(n) \leq n$.

We show inductively that for each integers $r \geq 0$ there exists an integer $s \geq 1$ with $f^s(n) = f(n) + r$. For $r = 0$ take $s = 1$. Induction step: Suppose that $f^s(n) = f(n) + r$ for some integer $s \geq 1$. Denote $t = f^{s-1}(n)$ and note that $f(n) + r + 1 = f^s(n) + 1 = f(t) + 1 = f^{g(t)}(t) = f^{g(t)+s-1}(n)$, hence $s' = g(t) + s - 1 \geq 1$ works for $r + 1$.

In particular, setting $r = n - f(n) \geq 0$ we see that $f^s(n) = n$ for some $s \geq 1$. This contradicts the previous claim. □

Now, for all positive integers m and n we obtain

$$f^m(n) \geq f^{m-1}(n) + 1 \geq \ldots \geq f(n) + m - 1.$$

Set $m = g(n)$. We obtain $f(n) + 1 = f^{g(n)}(n) \geq f(n) + g(n) - 1$, hence $g(n) \leq 2$ for all n. This contradicts the assumption that g is unbounded.

Now, let $k \geq 2$. We construct f and g satisfying desired conditions.

Let $n_1 < n_2 < n_3 < \dots$ be the sequence consisting of all positive integers not divisible by k (i.e. $n_{i(k-1)+j} = ik + j$ for any $i \geq 0$, $j \in \{1, 2, \dots, k-1\}$). Consider the sequence

$$k, n_1, 2k, n_2, n_3, n_4, 3k, n_5, \dots, n_9, 4k, n_{10}, \dots, n_{16}, 5k, \dots, ik, n_{(i-1)^2+1}, \dots, n_{i^2}, (i+1)k, \dots.$$

Note that every positive integer occurs in this sequence exactly once and for every n the number $n+k$ appears after n. For every n let $f(n)$ be the successor of n in this sequence and let $g(n)$ be the number of terms in this sequence between $f(n)$ and $f(n)+k$ (inclusive — we count $f(n)$ and $f(n)+k$ as well). By previous remarks, f and g are well defined and satisfy $f^{g(n)}(n) = f(n)+k$. Moreover, $g(n_{i^2}) = 2i + 3$ for any i, hence g is unbounded.

Solution 2. We will prove that if $k = 1$ then g is necessarily bounded.

The given equation implies that if $m = f(n)$ is in the image of f then $m + 1 = f(n) + 1 = f(f^{g(n)-1}(n))$ is in the image of f as well. Let $f(a)$ be the minimum of the image of f. Then the image of f is equal to $\{f(a), f(a) + 1, \dots\}$.

However, an easy inductive argument shows that for every m the number $f(a) + m$ is of the form $f^n(a)$. Hence the set $\{f(a), f^2(a), \dots\}$ is also equal to the image of f.

If $f^x(a) = f^y(a)$ for some $x > y$, then the sequence $a_n = f^n(a)$ is eventually periodic with a period $x - y$, but then the set $\{f(a), f^2(a), \dots\}$ is finite, which is a contradiction.

Therefore, for every $n \in \mathbb{N}_0$, there exists a unique positive integer x_n such that $f^{x_n}(a) = n+f(a)$, and conversely, for all $x \geq 1$ the number $f^x(a)$ is of the form $m + f(a)$ for some $m \geq 0$. In other words, the map $\mathbb{N}_0 \ni n \mapsto x_n \in \mathbb{N}$ is bijective.

Furthermore, $f^{x_{n+1}}(a) = (f(a)+n)+1 = f^{x_n}(a)+1 = f(f^{x_n-1}(a))+1 = f^{g(f^{x_n-1}(a))}(f^{x_n-1}(a)) = f^{x_n-1+g(f^{x_n-1}(a))}(a)$, which implies $x_{n+1} = x_n - 1 + g(f^{x_n-1}(a)) > x_n$, since $g(t) > 1$ for all t.

Therefore, the map $n \mapsto x_n$ is a strictly increasing bijection from $\mathbb{N}_0$ to $\mathbb{N}$ which gives $x_n = n+1$ for all n. Thus $f^{n+1}(a) = f(a) + n$, which implies $f(f(n)) = f(n) + 1$ for all $n \in \mathbb{N}$, hence $g(n) = 2$ for all $n \in \mathbb{N}$. Obviously, this means g is bounded.

Suppose now that $k \geq 2$. We will give an explicit example of functions f and g satisfying required properties.

- For each positive integer n, let $f(k^n) = nk + 1$ and let $f(nk + 1) = k^n + 2$.
- For each positive integer a which is not a power of k, let $f(ak) = ak + 2$.
- For any other positive integer x, let $f(x) = x + 1$.

In other words: We take the sequence of all positive integers 1, 2, 3, ..., remove from it all numbers congruent to 1 modulo k except for 1 itself, then insert them again a bit later, with each $nk+1$ occurring directly after k^n. That is, we get the sequence (shown here for a $k \geq 4$)

$$1, 2, \dots, k, k{+}1, k{+}2, \dots, 2k, 2k{+}2, \dots, 3k, 3k{+}2, \dots, k^2, 2k{+}1, k^2{+}2, \dots, k^3, 3k{+}1, k^3{+}2, \dots,$$

and for every n we let $f(n)$ be the successor of n in this sequence.

For example in the case $k = 2$, we get the sequence

$$1, \mathbf{2}, 3, \mathbf{4}, 5, 6, \mathbf{8}, 7, 10, 12, 14, \mathbf{16}, 9, 18, 20, \dots, \mathbf{32}, 11, 34 \dots$$

and in the case $k = 5$, we get the sequence

$$1, 2, 3, 4, \mathbf{5}, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 19, 20, 22, 23, 24, \mathbf{25}, 11, 27, 28, 29, 30, 32, 33, \dots,$$

with the powers of k highlighted in each case.

Like in the first solution we note that for every n the number $n+k$ appears after n, so we again define $g(n)$ as the number of terms in the sequence between $f(n)$ and $f(n)+k$ (inclusively), or equivalently, define $g(n)$ to be the smallest m such that $f^m(n) = f(n) + k$.

Now, $g(k^n)$ (denoting the distance between $f(k^n) = nk + 1$ and $f(k^n) + k = nk + 1 + k = (n + 1)k + 1 = f(k^{n+1})$ which occur after k^n and k^{n+1}, respectively) is equal to two plus the number of integers i such that $k^n + 2 \leq i \leq k^{n+1}$ and i is not congruent to 1 modulo k. For easier calculation we note that this is equivalent to the amount of integers i with $k^n < i \leq k^{n+1}$ and i not congruent to 1 modulo k, and observe that there exist exactly

$$(k^{n+1} - k^n) \cdot \frac{k-1}{k} = (k^n - k^{n-1}) \cdot (k - 1) = k^{n-1} \cdot (k - 1)^2$$

such numbers, which grows arbitrarily large for sufficiently large n. Therefore, g is also unbounded.

A note on possible constructions. The two shown constructions share a common idea that can be generalized further: Taking the sequence of all positive integers, removing any one residue class modulo k, and sprinkling it back in at increasingly large distances will always result in functions f and g with the desired properties.

I-2

We call a positive integer N *contagious* if there exist 1000 consecutive non-negative integers such that the sum of all their digits is N. Find all contagious positive integers.

Answer. All $N \geq 13500$.

Any solution naturally splits into two parts:

Part 1: Showing that no $N < 13500$ is contagious.

Part 2: Showing that all $N \geq 13500$ are contagious.

We present one approach to Part 1 and three approaches to Part 2 (by direct construction, by induction, and by discrete continuity).

Part 1. We make the following observation:

(T) Consider a block of 1000 consecutive non-negative integers. Then the last three digits of those numbers (prepended by zeros if needed) form a set $\{000, 001, \ldots, 999\}$.

Thus, given any such block, the sum of the last three digits alone equals $3 \cdot 100 \cdot (0+1+\cdots+9) = 13500$ (since each of the digits $0, 1, \ldots, 9$ occurs 100 times in each of the 3 positions). Therefore no integer less than 13500 is contagious.

Part 2, by direct construction. Fix $N \geq 13500$ and write the "remaining" digit sum as $N - 13500 = d \cdot 1000 + r$, where $d \geq 0$ and $r \in [0, 999]$ are non-negative integers. Write $r = \overline{r_2 r_1 r_0}$ as a 3-digit number (prepended by zeros if needed). Consider a number

$$X = \overline{\underbrace{11\ldots1}_{d \text{ times}} r_2 r_1 r_0}$$

formed by concatenating d copies of the digit 1 and the digits r_2, r_1, r_0. (If $d = 0$ set $X = r$.) We claim that the total digit sum of the 1000 consecutive non-negative integers $X, X+1, \ldots, X+999$ equals N. Note that:

(a) Ignoring the last three digits, the $1000 - r$ numbers $X, \ldots, X + (999 - r)$ have digit sum $d \cdot 1 = d$ each and the next r numbers $X + (1000 - r), \ldots, X + 999$ have digit sum $(d-1) \cdot 1 + 2 = d + 1$ each.

(b) As in Part 1, the last three digits of all the 1000 numbers add up to 13500.

Therefore, all in all, we obtain that the total digit sum of $X, X+1, \dots, X+999$ equals

$$(1000-r)\cdot d + r\cdot(d+1) + 13500 = 1000d + r + 13500 = N,$$

as required.

Part 2, by induction. Given a non-negative integer n, denote by s_n the digit sum of n and by $S(n)$ the total digit sum of $n, n+1, \dots, n+999$, that is,

$$S(n) = s_n + s_{n+1} + \cdots + s_{n+999}.$$

We proceed by induction. As a first step, we show that the 1000 numbers $N \in \{13500, \dots, 14499\}$ are all contagious. As a second step, we show that if N is contagious, then $N+1000$ is contagious. Combined, this implies that all $N \geq 13500$ are contagious.

For the first step, note that for any integer $n \geq 0$ we have

$$S(n+1) - S(n) = (s_{n+1} + \cdots + s_{n+1000}) - (s_n + \cdots + s_{n+999}) = s_{n+1000} - s_n.$$

Thus, for $0 \leq X \leq 999$, we have $S(X+1) = S(X)+1$, since the number $X+1000$ has an extra digit 1 in front of the (up to three-digit) number X. Since $S(0) = 13500$ by Part 1, we get $S(X) = 13500 + X$ for $0 \leq X \leq 999$. Therefore all $N \in \{13500, \dots, 14499\}$ are indeed contagious.

For the second step, suppose that N is contagious, that is, there exist 1000 consecutive integers $X, X+1, \dots, X+999$ with total digit sum N. Take any integer i such that $10^i > X + 999$. Then the 1000 consecutive integers

$$10^i + X, 10^i + X + 1, \dots, 10^i + X + 999$$

have a total digit sum equal to $N + 1000$ (since each number got an extra digit 1 and, possibly, several zeroes).

Part 2, by discrete continuity. We make three observations:

(A) For any integer $n \geq 0$, we have $S(n+1) - S(n) \leq 1$.

- Indeed, as before, we have

  $$S(n+1) - S(n) = (s_{n+1} + \cdots + s_{n+1000}) - (s_n + \cdots + s_{n+999}) = s_{n+1000} - s_n.$$

  Note that the numbers $n+1000$ and n have the same last three digits. We distinguish two cases:

a) If the fourth digit of n from the right is less than 9, then the digits of $n + 1000$ and n differ only in that position and we have $s_{n+1000} - s_n = 1$. (If n is 3-digit, this is true too.)

b) Otherwise, suppose that there are $d \geq 1$ consecutive digits 9 just in front of the last three digits of n. Then $s_{n+1000} - s_n = 1 - 9d < 1$, because the resulting number will have d zeroes in place of the nines, and the digit to the left of the nines increased by one.

(B) We have $S(0) = 13500$.

- By the same argument as in Part 1 we get $S(0) = 3 \cdot 100 \cdot (0 + 1 + \cdots + 9) = 13500$.

(C) The sequence $S(n)$ is unbounded as $n \to \infty$.

- For instance, setting $n = 10^k - 1$ we get $S(n) \geq s_n = 9k$.

It remains to put the observations together. By (B), the number $n = 13500$ is contagious. Now fix $N \geq 13501$. Since the sequence $(S(n))_{n=0}^{\infty}$ is unbounded, there exists an integer $k \geq 1$ such that $S(k) \geq N$. Take the smallest such k. By minimality of k we have $S(k-1) \leq N - 1$. Combining this with (A) we now deduce

$$N \leq S(k) \leq 1 + S(k-1) \leq 1 + (N-1) = N,$$

hence $S(k) = N$ implying that N is contagious.

I-3

Let ABC be an acute scalene triangle with circumcircle ω and incenter I. Suppose the orthocenter H of BIC lies inside ω. Let M be the midpoint of the longer arc BC of ω. Let N be the midpoint of the shorter arc AM of ω.

Prove that there exists a circle tangent to ω at N and tangent to the circumcircles of BHI and CHI.

Solution 1. Denote the circumcircles of BHI and CHI by ω_1 and ω_2 and their centers by O_1 and O_2, respectively. Let O be the center of ω. Let R be the radius of ω.

Since H is the orthocenter of triangle BIC it follows that I is the orthocenter of triangle BHC. Therefore

$$\angle HIB = 180^\circ - (\angle BHI + \angle IBH) = 180^\circ - (90^\circ - \angle CBH + 90^\circ - \angle BHC) = 180^\circ - \angle HCB,$$

and analogously we get $\angle CIH = 180^\circ - \angle CBH$ and $\angle BIC = 180^\circ - \angle BHC$.

Denote by r the radius of circle ω_1, then from sine law we get

$$2r = \frac{HB}{\sin \angle HIB} = \frac{HB}{\sin(180^\circ - \angle HIB)} = \frac{HB}{\sin \angle HCB} = \\ = \text{diameter of circumcircle of the triangle } BHC.$$

Using the same argument for triangles CIH and BIC we see that r is equal to radii of ω_1, ω_2, circumcircles of BIC and BHC.

From the following angle chase it follows that

$$\angle BHC = 180^\circ - \angle BIC = 180^\circ - \left(180^\circ - \frac{1}{2}\angle CBA - \frac{1}{2}\angle BCA\right) = \\ = \frac{1}{2}(\angle CBA + \angle BCA) = 90^\circ - \frac{1}{2}\angle BAC.$$

Since H lies inside ω and $\angle BAC$ is acute we conclude that

$$\angle BAC < \angle BHC = 90^\circ - \frac{1}{2}\angle BAC < 90^\circ$$

so

$$2r = \text{diameter of circumcircle of } BHC = \frac{BC}{\sin \angle BHC} < \frac{BC}{\sin \angle BAC} = 2R,$$

thus $r < R$.

Let $\angle BAC = \alpha$, $\angle CBA = \beta$, $\angle ACB = \gamma$. Then

$$\angle BO_1I = 2\angle BHI = 2(90^\circ - \angle CBH) = 2\angle ICB = \gamma,$$

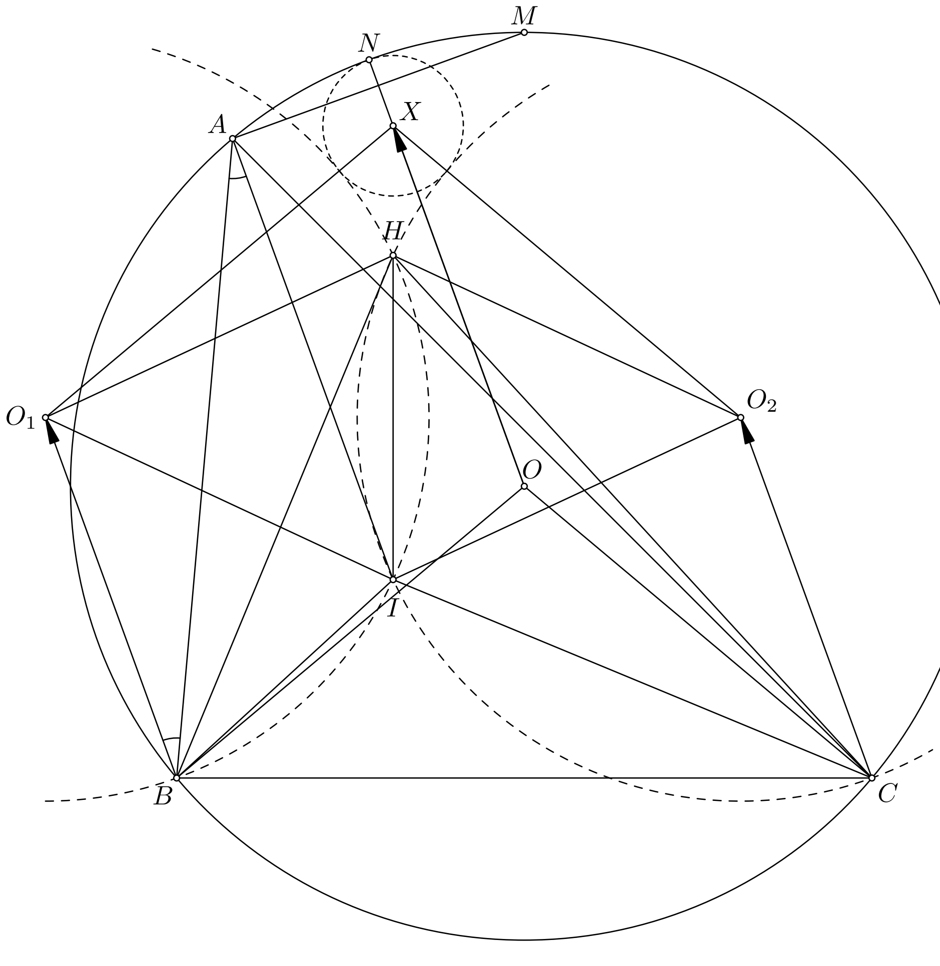
so

$$\angle IBO_1 = 90^\circ - \frac{1}{2}\angle BO_1I = 90^\circ - \frac{\gamma}{2} = \frac{\alpha+\beta}{2},$$

and finally

$$\angle ABO_1 = \angle IBO_1 - \angle IBA = \frac{\alpha+\beta}{2} - \frac{\beta}{2} = \frac{\alpha}{2} = \angle BAI.$$

This shows that $BO_1 \parallel AI$, and moreover, rays $BO_1^{\rightarrow}$, $AI^{\rightarrow}$ determine opposite directions. Similarly, rays $CO_2^{\rightarrow}$, $AI^{\rightarrow}$ are parallel and determine opposite directions. Therefore these rays are parallel and $BO_1^{\rightarrow}$, $CO_2^{\rightarrow}$ determine the same direction. Since $BO_1 = r = CO_2$, it follows that vectors $\overrightarrow{BO_1}$, $\overrightarrow{CO_2}$ are equal. Denote this vector by $\overrightarrow{v}$.



Note that $ON \perp AM$. Moreover

$$\begin{aligned}\angle IAM &= \angle IAC + \angle CAM = \angle IAC + \angle CBM = \\ &= \angle IAC + \frac{1}{2}(180^\circ - \angle BMC) = \angle IAC + \frac{1}{2}(180^\circ - \angle BAC) = 90^\circ,\end{aligned}$$

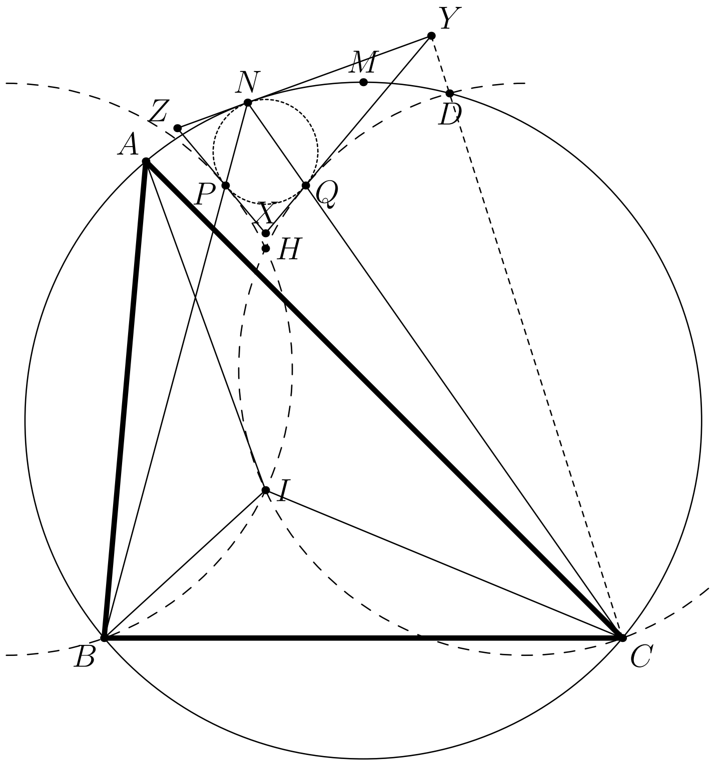
so $AM \perp AI$, hence $ON \parallel AI \parallel BO_1 \parallel \overrightarrow{v}$. Let X be a point such that $\overrightarrow{OX} = \overrightarrow{v}$. Since $ON \parallel \overrightarrow{v}$, X lies on line ON. It actually lies on ray $ON^{\rightarrow}$ since rays $ON^{\rightarrow}$, $AI^{\rightarrow}$ determine opposite directions.

Note that translation by $\overrightarrow{v}$ maps triangle BCO to triangle O_1O_2X. Therefore $O_1X = BO = R$ and $O_2X = CO = R$.

Let ω' be the circle centered at X with radius $R - r > 0$.

Observe that $O_1X = R = r + (R - r)$, so ω' is tangent externally to ω_1. For similar reason it is tangent externally to ω_2. Moreover $OX = r = R - (R - r) = ON - XN$, so ω' is tangent to ω internally at point N.

Solution 2. Let $\beta := \angle ABI = \angle IBC$ and $\gamma := \angle BCI = \angle ICA$. Without loss of generality, we assume that $\beta > \gamma$. Furthermore, we define P to be the intersection of NB and the circumcircle of BHI and Q to be the intersection of NC and the circumcircle of CHI.



Our goal is to prove that the circumcircle of NPQ satisfies the desired conditions. To this end, define the tangent t_N to ω through N, tangent t_P to the circumcircle of BHI through P and tangent t_Q to the circumcircle of CHI through Q. Let X be the intersection of t_P and t_Q, Y the intersection of t_N and t_Q and Z the intersection of t_N and t_P. If we can prove that $PX = QX$, $NY = QY$ and $NZ = PZ$, it follows that the circumcircle of NPQ is the incircle of XYZ and N, P and Q are the contact points. By definition of t_N, t_P and t_Q, this would imply that the circumcircle of NPQ satisfies the desired properties.

Using that the arcs MN and NA have the same lengths and the triangle BMC is isosceles, we calculate some angles:

$$\begin{aligned}\angle ACN = \angle ABN &= \frac{1}{2}\angle ABM \\ &= \frac{1}{2}(\angle ABC - \angle MBC) \\ &= \frac{1}{2}\left(2\beta - \frac{180^\circ - \angle CMB}{2}\right) \\ &= \frac{1}{2}\left(2\beta - \frac{180^\circ - (180^\circ - 2\beta - 2\gamma)}{2}\right) \\ &= \frac{1}{2}(\beta - \gamma).\end{aligned}$$

Furthermore, using that H is the orthocenter of BIC and has to lie outside of it but inside ω, we compute

$$\begin{aligned}\angle ICQ &= \angle ICA + \angle ACN = \gamma + \frac{1}{2}(\beta - \gamma) = \frac{1}{2}(\beta + \gamma), \\ \angle PBI &= \angle ABI - \angle ABN = \beta - \frac{1}{2}(\beta - \gamma) = \frac{1}{2}(\beta + \gamma), \\ \angle BCH &= 90^\circ - \angle IBC = 90^\circ - \beta \\ \angle HIQ &= \angle HCQ \\ &= \angle BCI + \angle ICQ - \angle BCH \\ &= \gamma + \frac{1}{2}(\beta + \gamma) - (90^\circ - \beta) \\ &= \frac{1}{2}(3\beta + 3\gamma - 180^\circ).\end{aligned}$$

By an analogous computation, we find that

$$\angle PIH = \frac{1}{2}(3\beta + 3\gamma - 180^\circ).$$

If we now define X' as the intersection of t_P and HI as well as X'' the intersection of t_Q and HI, we obtain by tangency:

$$\angle X'PI = \frac{1}{2}\angle PBI = \frac{1}{2}(\beta + \gamma) = \frac{1}{2}\angle ICQ = \angle IQX''.$$

Since also

$$\angle PIX' = \angle PIH = \frac{1}{2}(3\beta + 3\gamma - 180^\circ) = \angle HIQ = \angle X''IQ,$$

we get two similar triangles $X'PI$ and $X''QI$. Also, since HI is the power line of the circumcircles of BHI and CHI, we have

$$X'P^2 = X'H \cdot X'I, \quad X''Q^2 = X''H \cdot X''I,$$

and hence

$$\frac{X'H \cdot X'I}{X''H \cdot X''I} = \left(\frac{X'P}{X''Q}\right)^2 = \left(\frac{X'I}{X''I}\right)^2,$$

using similarity. By simplifying those terms, we get

$$\frac{X'H}{X''H} = \frac{X'I}{X''I} = \frac{X'H + HI}{X''H + HI},$$

hence $X'H \cdot HI = X''H \cdot HI$ and therefore $X'H = X''H$, which implies that $X' = X'' = X$. By the power of X to BHI and CHI, we finally get $XP^2 = XQ^2$, so $XP = XQ$ as desired.

To see that $NY = QY$, define D as the second intersection of the circumcircles of CHI and ABC and let Y' be the intersection of CD and t_N. We want to show that $Y' = Y$. Using that $\angle CQI = \angle CHI = 90° - \angle BCH = 90° - (90° - \angle IBC) = \beta$, we compute

$$\begin{aligned}\angle QDY' &= 180° - \angle CDQ \\ &= \angle QIC \\ &= 180° - \angle CQI - \angle ICQ \\ &= 180° - \beta - \frac{1}{2}(\beta + \gamma).\end{aligned}$$

On the other hand, the tangency to ω at N yields

$$\angle Y'NQ = \angle Y'NC = \angle NBC = \angle IBC + \angle NBI = \beta + \frac{1}{2}(\beta + \gamma).$$

Now, since $\angle QDY' + \angle Y'NQ = 180°$, we conclude that $NQDY'$ is a cyclic quadrilateral. The same is true for $NQDY$ because of

$$\angle YND = \angle NCD = \angle QCD = \angle YQD,$$

where we used tangency of t_N to ω and of t_Q to the circumcircle of CHI. Since both Y and Y' lie on t_N, they have to be the same point. Since CD is the power line of circle ω and the circumcircle of CHI, we obtain $YQ^2 = YN^2$, so $QY = NY$. We can find a completely analogous argument for $PZ = NZ$ to conclude.

I-4

Find all positive integers n for which there exist positive integers x_1, x_2, …, x_n such that

$$\frac{1}{x_1^2}+\frac{2}{x_2^2}+\frac{4}{x_3^2}+\cdots+\frac{2^{n-1}}{x_n^2}=1.$$

Answer. Solutions exist for all positive integers n except for $n=2$.

Solution 1.

- $n=1$:

  Here, $x_1 := 1$ provides a solution, since

  $$\frac{1}{1^2}=1\,.$$

- $n=2$:

  Here, no solution exists. Indeed, $x_1=1$ or $x_2=1$ yields $\frac{1}{x_1^2}+\frac{2}{x_2^2}>1$, while $x_1,\ x_2\geq 2$ leads to

  $$\frac{1}{x_1^2}+\frac{2}{x_2^2}\leq\frac{1}{4}+\frac{2}{4}=\frac{3}{4}<1\,.$$

- $n=4$:

  Here, $(x_1,x_2,x_3,x_4):=(3,3,3,6)$ provides a solution, since

  $$\frac{1}{3^2}+\frac{2}{3^2}+\frac{4}{3^2}+\frac{8}{6^2}=\frac{7}{9}+\frac{8}{36}=\frac{7}{9}+\frac{2}{9}=1\,.$$

- Induction step from n to $(n+2)$:

  Let $(y_1,y_2,\ldots,y_n)$ be a solution for n, i.e.,

  $$\frac{1}{y_1^2}+\frac{2}{y_2^2}+\frac{4}{y_3^2}+\cdots+\frac{2^{n-1}}{y_n^2}=1\,.$$

  Then

  $$(x_1,x_2,\ldots,x_{n+2}):=(2,2,4y_1,4y_2,\ldots,4y_n)$$

is a solution for $(n+2)$, since

$$\begin{aligned}\frac{1}{x_1^2}+\frac{2}{x_2^2}+\frac{4}{x_3^2}+\cdots+\frac{2^{n+1}}{x_{n+2}^2} &= \frac{1}{2^2}+\frac{2}{2^2}+\frac{4}{(4y_1)^2}+\cdots+\frac{2^{n+1}}{(4y_n)^2}\\ &= \frac{1}{4}+\frac{2}{4}+\frac{4}{16}\left[\frac{1}{(y_1)^2}+\cdots+\frac{2^{n-1}}{(y_n)^2}\right]\\ &= \frac{3}{4}+\frac{1}{4}\cdot 1\\ &= 1\,.\end{aligned}$$

Using this induction step and the solutions for $n = 1$ and $n = 4$, we can construct solutions for all $n \geq 3$.

Solution 1a. There are other induction steps possible. For example from n to $(n+3)$:

Let $(y_1, y_2, \ldots, y_n)$ be a solution for n, i.e.,

$$\frac{1}{y_1^2}+\frac{2}{y_2^2}+\frac{4}{y_3^2}+\cdots+\frac{2^{n-1}}{y_n^2}=1\,.$$

Then

$$(x_1, x_2, \ldots, x_{n+3}) := (3, 3, 3, 6y_1, 6y_2, \ldots, 6y_n)$$

is a solution for $(n+3)$, since

$$\begin{aligned}\frac{1}{x_1^2}+\frac{2}{x_2^2}+\frac{4}{x_3^2}+\cdots+\frac{2^{n+2}}{x_{n+3}^2} &= \frac{1}{3^2}+\frac{2}{3^2}+\frac{4}{3^2}+\frac{8}{(6y_1)^2}+\cdots+\frac{2^{n+2}}{(6y_n)^2}\\ &= \frac{1}{9}+\frac{2}{9}+\frac{4}{9}+\frac{8}{36}\left[\frac{1}{(y_1)^2}+\cdots+\frac{2^{n-1}}{(y_n)^2}\right]\\ &= \frac{7}{9}+\frac{2}{9}\cdot 1\\ &= 1\,.\end{aligned}$$

In order to complete this approach, of course, solutions have to be provided for $n = 1$, 3, and 5.

In fact, every solution $(z_1, \ldots, z_k)$ for k yields an induction step from n to $(n+k-1)$. Indeed, if $(y_1, \ldots, y_n)$ is a solution then

$$(z_1, \ldots, z_{k-1}, z_k \cdot y_1, \ldots, z_k \cdot y_n)$$

is a solution, too. The two constructions presented above belong to $(2, 2, 4)$ and $(3, 3, 3, 6)$.

So it is conceivable that someone finds an induction from n to, say, $(n+6)$. In this case, solutions for six suitable small values of n would be necessary in order to complete the approach.

Solution 2. As in the previous solution, we can show that there does not exist a solution for $n = 2$ and find explicit solutions for $n = 1$ and $n = 3$.

We construct further solutions by induction. Let $(y_1, y_2, \ldots, y_n)$ be a solution for $n \geq 3$.

Then setting $x_{n-2} = x_{n+1} \coloneqq 3y_{n-2}$ and $x_i \coloneqq y_i$ for all other i is a solution for $n + 1$, since for the sum of the terms corresponding to x_{n-2} and x_{n+1} (which are the only ones that differ between the solutions for n and $n + 1$) we get

$$\frac{2^{n-3}}{x_{n-2}^2} + \frac{2^n}{x_{n+1}^2} = \frac{2^{n-3}}{9y_{n-2}^2} + \frac{2^n}{9y_{n-2}^2} = \frac{(1+8)\cdot 2^{n-3}}{9y_{n-2}^2} = \frac{2^{n-3}}{y_{n-2}^2},$$

thereby keeping the sum of all terms equal.

Solution 3. As in the other solutions, we can show that there does not exist a solution for $n = 2$. Also, we can find explicit solutions for $n = 4$, $n = 6$ and $n = 8$. Now we prove that we can find suitable integers for all other n:

For $n = 2k + 1$ (where k is a suitable non-negative integer), we can choose

$$x_1 = \cdots = x_{n-1} = 2^k, \quad x_n = 2^{2k}$$

to obtain

$$\sum_{i=1}^{n} \frac{2^{i-1}}{x_i^2} = \sum_{i=1}^{2k} \frac{2^{i-1}}{2^{2k}} + \frac{2^{2k}}{2^{4k}} = \frac{\frac{2^{2k}-1}{2-1}}{2^{2k}} + \frac{1}{2^{2k}} = 1.$$

For $n = 2k$ with an integer $k \geq 5$, observe that, by setting $x_i = 2^{\frac{i-1}{2}}$ for odd i and $x_i = 2^{\frac{i}{2}-1}$ for even i, we have

$$\sum_{i=1}^{2k} \frac{2^{i-1}}{x_i^2} = \sum_{j=1}^{k} \left(\frac{2^{2j-2}}{2^{2j-2}} + \frac{2^{2j-1}}{2^{2j-2}} \right) = \sum_{j=1}^{k} (1+2) = 3k.$$

If we can modify the x_i in order to obtain some square number m^2 on the right hand side, we can in a second step multiply each x_i by m to obtain a sum of 1.

Observe that all x_i for $i > 2$ are even. We show that there is a square divisible by 3 that can be obtained by replacing some of the x_i (with $i > 2$) by $x_i' \coloneqq x_i/2$:

Note that for odd i, this increases the sum by $4 - 1 = 3$ and for even i, this increases the sum by $8 - 2 = 6$. Since there are $k - 1$ odd and $k - 1$ even indexes to choose from, we can increase the sum by every number l that is divisible by three and satisfies $0 \leq l \leq 3(k-1) + 6(k-1) = 9k - 9$.

Because of $k \geq 5$ and therefore $\sqrt{3k} < k$, the smallest square $m^2 \geq 3k$ which is divisible by 3 certainly satisfies

$$(\sqrt{3k})^2 = 3k \leq m^2 \leq (\sqrt{3k} + 3)^2 = 3k + 6\sqrt{3k} + 9 < 9k + 9.$$

In particular, $m^2 \leq 9k+6$ because m is divisible by 3. This means that in order to increase $3k$ to m^2, we have to add a number between 0 and $6k+6$ to the sum above. However, $6k+6 \leq 9k-9$ because $k \geq 5$ and by the above argument, we can always do that.

To summarize, we first set the x_i as above, then select up to $2k-2$ of them to be divided by 2, then multiply all of them by m, yielding a right hand side of 1.