

EGMO2022 Problems

DAY 1

Problem 1.

Let ABC be an acute-angled triangle in which $BC < AB$ and $BC < CA$. Let point P lie on segment AB and point Q lie on segment AC such that $P \neq B$, $Q \neq C$ and $BQ = BC = CP$. Let T be the circumcentre of triangle APQ , H the orthocentre of triangle ABC , and S the point of intersection of the lines BQ and CP . Prove that T , H and S are collinear.

Problem 2.

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of all positive integers. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for any positive integers a and b , the following two conditions hold:

- (1) $f(ab) = f(a)f(b)$, and
- (2) at least two of the numbers $f(a)$, $f(b)$ and $f(a+b)$ are equal.

Problem 3.

An infinite sequence of positive integers a_1, a_2, \dots is called *good* if

- (1) a_1 is a perfect square, and
- (2) for any integer $n \geq 2$, a_n is the smallest positive integer such that

$$na_1 + (n-1)a_2 + \dots + 2a_{n-1} + a_n$$

is a perfect square.

Prove that for any good sequence a_1, a_2, \dots , there exists a positive integer k such that $a_n = a_k$ for all integers $n \geq k$.

EGMO2022 Problems

DAY 2

Problem 4.

Given a positive integer $n \geq 2$, determine the largest positive integer N for which there exist $N + 1$ real numbers a_0, a_1, \dots, a_N such that

$$(1) \quad a_0 + a_1 = -\frac{1}{n}, \text{ and}$$

$$(2) \quad (a_k + a_{k-1})(a_k + a_{k+1}) = a_{k-1} - a_{k+1} \text{ for } 1 \leq k \leq N - 1.$$

Problem 5.

For all positive integers n, k , let $f(n, 2k)$ be the number of ways an $n \times 2k$ board can be fully covered by nk dominoes of size 2×1 . (For example, $f(2, 2) = 2$ and $f(3, 2) = 3$.)

Find all positive integers n such that for every positive integer k , the number $f(n, 2k)$ is odd.

Problem 6.

Let $ABCD$ be a cyclic quadrilateral with circumcentre O . Let the internal angle bisectors at A and B meet at X , the internal angle bisectors at B and C meet at Y , the internal angle bisectors at C and D meet at Z , and the internal angle bisectors at D and A meet at W . Further, let AC and BD meet at P . Suppose that the points X, Y, Z, W, O and P are distinct.

Prove that O, X, Y, Z and W lie on the same circle if and only if P, X, Y, Z and W lie on the same circle.

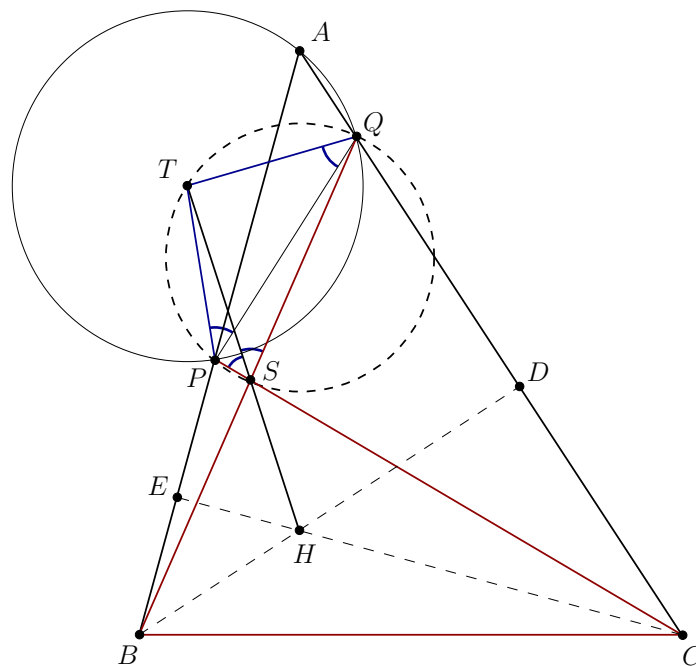
EGMO2022 Problems – Solutions

Problem 1.

Let ABC be an acute-angled triangle in which $BC < AB$ and $BC < CA$. Let point P lie on segment AB and point Q lie on segment AC such that $P \neq B$, $Q \neq C$ and $BQ = BC = CP$. Let T be the circumcentre of triangle APQ , H the orthocentre of triangle ABC , and S the point of intersection of the lines BQ and CP . Prove that T , H and S are collinear.

Proposed by: Netherlands

Solution 1.



We show that T and H are both on the angle bisector ℓ of $\angle BSC$.

We first prove that $H \in \ell$. The altitude CH in triangle ABC is also the altitude in isosceles triangle PBC with $CP = CB$. Therefore, CH is also the angle bisector of $\angle PCB$ and hence also of $\angle SCB$. Analogously, BH is the angle bisector of $\angle SBC$. We conclude that H , as the intersection of two angle bisectors of $\triangle BSC$, is also on the third angle bisector, which is ℓ .

We now prove that $T \in \ell$.

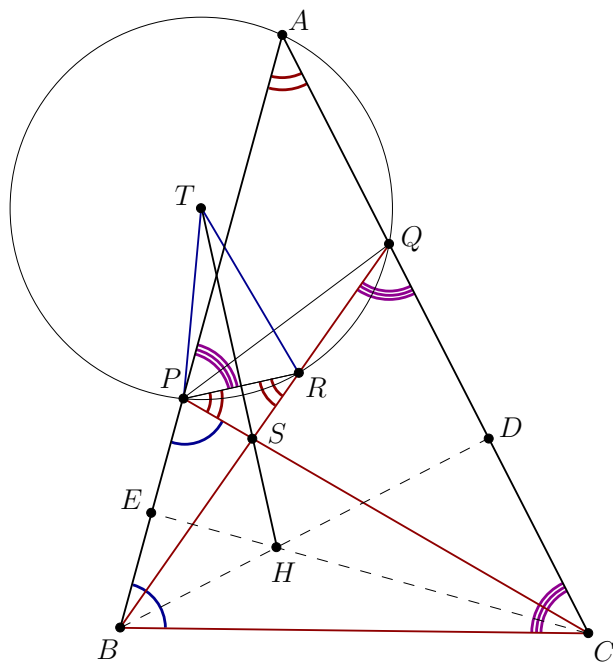
Variant 1. In the isosceles triangles $\triangle BCP$ and $\triangle CBQ$ we see that $\angle BCP = 180^\circ - 2\angle B$ and $\angle CBQ = 180^\circ - 2\angle C$. This yields $\angle PSQ = \angle BSC = 180^\circ - (180^\circ - 2\angle B) - (180^\circ - 2\angle C) = 180^\circ - 2\angle A$. Furthermore, $\angle PTQ = 2\angle PAQ = 2\angle A$ (T being circumcentre of $\triangle APQ$). Now $\angle PTQ + \angle PSQ = 180^\circ$, so $PTQS$ is a cyclic quadrilateral. From $PT = TQ$ we then obtain that $\angle PST = \angle PQT = \angle QPT = \angle QST$, so T is on the angle bisector $\angle PSQ$, which is also ℓ .

We conclude that T , S and H are collinear.

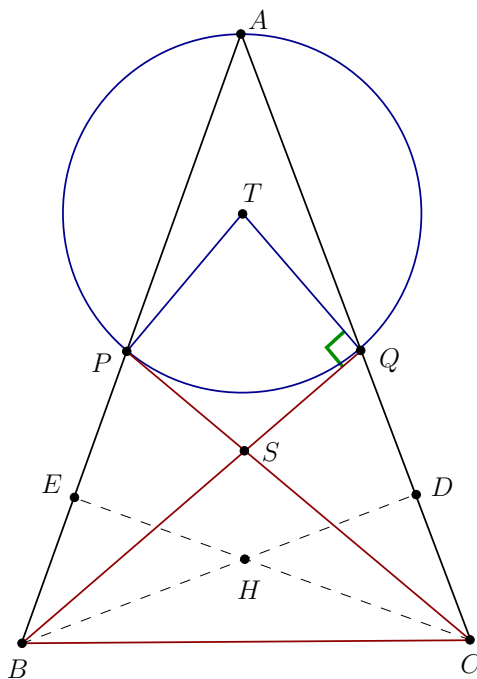
Variant 2. Let R be the second intersection of BQ and $\odot APQ$.

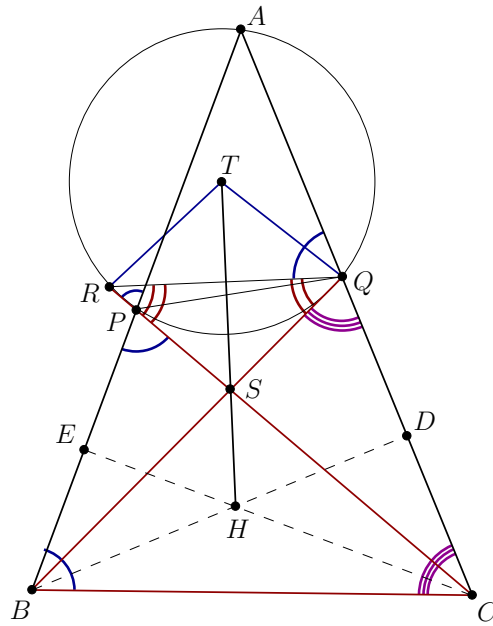
$APRQ$ is cyclic quadrilateral, so $\angle PRS = \angle A$, $\angle APR = \angle BQC = \angle C$. On the other hand $\angle BPC = \angle B$. Therefore $\angle RPS = 180^\circ - \angle B - \angle C = \angle A$. Hence, the triangle PRS is isosceles with $SP = SR$; then ℓ is the perpendicular bisector of the chord PR in the circle that passes through T .

Note that if BQ is tangent to $\odot APQ$, then CP is also tangent to $\odot APQ$ and triangle ABC is isosceles, so T , S and H lie on the altitude from A .

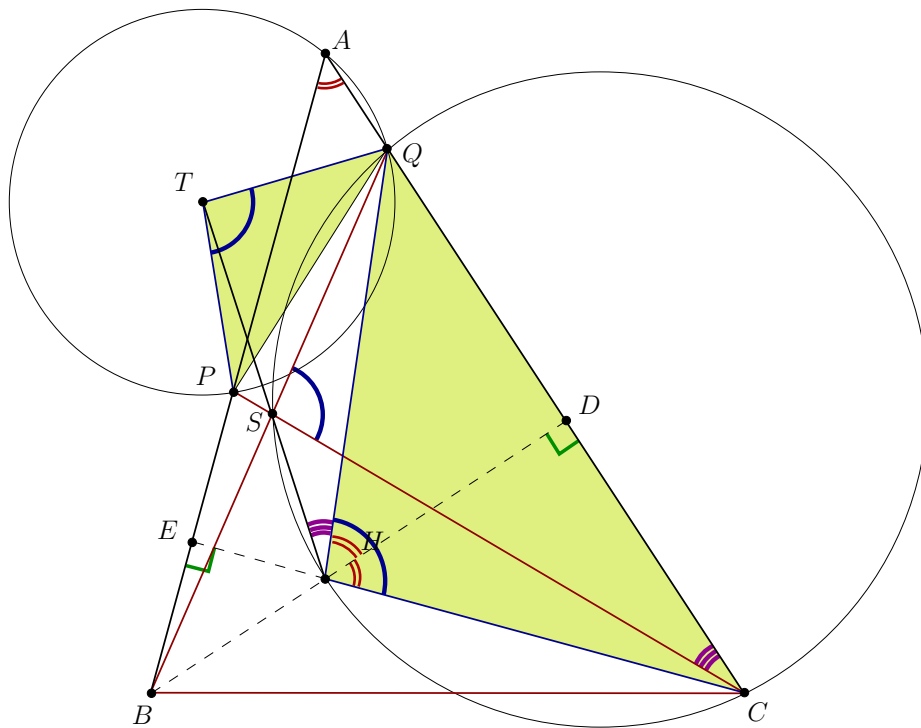


Remark. The fact that CP is also tangent to $\odot APQ$ could be shown with $PTQS$ being a cyclic quadrilateral like in the first variant of the solution. Otherwise we can consider R the second intersection of CP and $\odot APQ$ and prove that triangle QRS is isosceles.





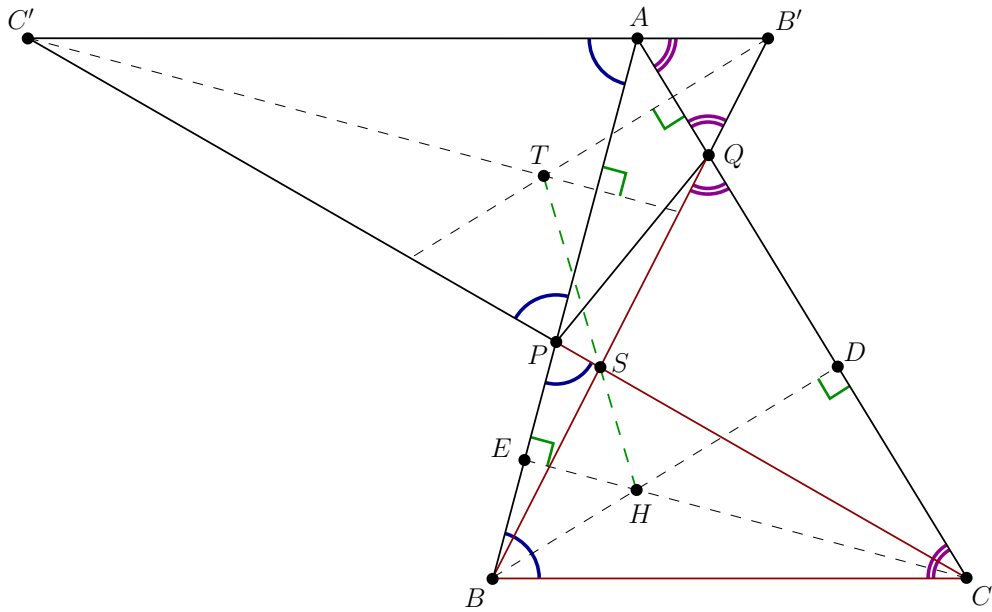
Solution 2.



In the same way as in the previous solution, we see that $\angle PSQ = 180^\circ - 2\angle A$, so $\angle CSQ = 2\angle A$. From the cyclic quadrilateral $AEHD$ (with E and D feet of the altitudes CH and BH) we see that $\angle DHC = \angle DAE = \angle A$. Since BH is the perpendicular bisector of CQ , we have $\angle DHQ = \angle A$ as well, so $\angle CHQ = 2\angle A$. From $\angle CHQ = 2\angle A = \angle CSQ$, we see $CHSQ$ is a cyclic quadrilateral. This means $\angle QHS = \angle QCS$.

Since triangles PTQ and CHQ are both isosceles with apex $2\angle A$, we get $\triangle PTQ \sim \triangle CHQ$. We see that one can be obtained from the other by a spiral similarity centered at Q , so we also obtain $\triangle QTH \sim \triangle QPC$. This means that $\angle QHT = \angle QCP$. Combining this with $\angle QHS = \angle QCS$, we see that $\angle QHT = \angle QCP = \angle QCS = \angle QHS$. So $\angle QHT = \angle QHS$, which means that T, S and H are collinear.

Solution 3.



Let us draw a parallel f to BC through A . Let $B' = BQ \cap f$, $C' = CP \cap f$. Then $AQB' \sim CQB$ and $APC' \sim BPC$, therefore both QAB' and APC' will be isosceles.

Also, $BCS \sim B'C'S$ with respect to the similarity with center S , therefore if we take the image of the line BH (which is the perpendicular bisector of the segment CQ) through this transformation, it will go through the point B' , and be perpendicular to AQ (as the image of CQ is parallel to CQ). As AQB' is isosceles, this line is the perpendicular bisector of the segment AQ . This means it goes through T , the circumcenter of APQ . Similarly on the other side the image of CH also goes through T . This means that the image of H with respect to the similarity though S will be T , so T, S, H are collinear.

Problem 2.

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of all positive integers. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for any positive integers a and b , the following two conditions hold:

- (1) $f(ab) = f(a)f(b)$, and
- (2) at least two of the numbers $f(a)$, $f(b)$ and $f(a + b)$ are equal.

Proposed by: Fedir Yudin, Ukraine

Answer: $f(n) = a^{v_p(n)}$, where $a \in \mathbb{N}$, p is a prime and $v_p(n)$ is the largest exponent of p dividing n .

Solution 1. First, all such functions f satisfy the conditions, as $v_p(a) \neq v_p(b)$ implies $v_p(a + b) = \min(v_p(a), v_p(b))$. Plugging $a, b = 1$ into (1) gives $f(1) = 1$. Also, a simple induction gives that $f(\prod_{i=1}^k p_i^{a_i}) = \prod_{i=1}^k f(p_i)^{a_i}$. Let S be the set of all primes p such that $f(p) \neq 1$. If $|S| = 1$, then $S = \{p\}$ and $f(p) = a$ for some $a \neq 1$, and thus $f(n) = a^{v_p(n)}$ for all n . If $S = \emptyset$ then $f(n) = 1$ for all n , which can be also written in the above form for $a = 1$.

Now suppose that S contains at least two primes. Let $p < q$ be the two smallest elements of S . Since all prime divisors of $q - p$ are smaller than q and not equal to p , $f(q - p) = 1$. Using (2) for p and $q - p$ gives us that some two of $f(p) \neq 1$, $f(q - p) = 1$ and $f(q) \neq 1$ are equal, meaning that $f(p) = f(q)$. Let $t \geq 2$ be the smallest integer such that $p^t > q$, and let $p^t = aq + b$ with $0 \leq b < q$. Since $q \geq p^{t-1}$ and q is not divisible by p , we have $q > p^{t-1}$, so $a < p$, and thus $f(a) = 1$. Therefore aq is not divisible by p , thus neither is b , and since $b < q$, $f(b) = 1$. Now using (2) for aq and b we get that some two of $f(aq) = f(a)f(q) = f(q) = f(p)$, $f(b) = 1$ and $f(aq + b) = f(p^t) = f(p)^t$ are equal, which is a contradiction.

Solution 1B (for the main part).

Suppose that there are two primes, $p < q$ such that $f(p) > 1$ and $f(q) > 1$, and these are the two smallest such primes.

$1 < p - q < q$ and it is not divisible by p and q , therefore its prime factors are smaller than q , so we conclude $f(q - p) = 1$. By (2), two of $f(q - p) = 1$, $f(q - p) > 1$ and $f(q) > 1$ are equal, so $f(q) = f(p) = a$.

If $p^2 < q$ then $q - p^2$ is not divisible by p and q , so we conclude $f(q - p^2) = 1$ in the same way. Then $f(p^2) = f(p)^2 = a^2$, $f(q - p^2) = 1$ and $f(q) = a$ are distinct, contradicting (2). Hence, $q < p^2$.

Let $c = \lfloor p^2/q \rfloor$; then $0 < c < p$, so $f(c) = 1$. By $0 < p^2 - cq < q$, we get $f(p^2 - cq) = 1$. Now $f(p^2 - cq) = 1$, $f(p^2) = a^2$ and $f(cq) = a$ are distinct, contradicting (2) again.

Solution 2 (for the main part).

Suppose there exists a positive integer n such that $f(n) \neq 1$ and $f(n + 1) \neq 1$. We know that $f(1) = 1$, so by (2) two of $f(n) \neq 1$, $f(n + 1) \neq 1$ and $f(1) = 1$ are equal, therefore $f(n) = f(n + 1)$.

$f(n^2) = f(n)^2 \neq 1$ and $f(n^2 - 1) = f(n + 1)f(n - 1) = f(n)f(n - 1) \neq 1$, but by (2) two of $f(1) = 1$, $f(n^2 - 1) \neq 1$ and $f(n^2) \neq 1$ are equal, so $f(n^2 - 1) = f(n^2)$ and thus $f(n)f(n - 1) = f(n)^2$, therefore $f(n - 1) = f(n)$.

Then for $n - 1$, it is also true that $f(n - 1) \neq 1$ and $f(n - 1 + 1) \neq 1$, so the same proof applies, $f(n - 1 - 1) = f(n - 1)$ and so on. By induction we get that $f(n) = f(1)$, which is a contradiction, as $f(1) = 1$ and we assumed that $f(n) \neq 1$. Therefore, there can't be any n for which $f(n) \neq 1$ and $f(n + 1) \neq 1$.

Now suppose there are two primes, $p \neq q \in S$ such that $f(p) \neq 1$ and $f(q) \neq 1$. Then by the Chinese remainder theorem, there exists a number n such that $p|n$ and $q|n + 1$, so $f(p)|f(n)$ and $f(q)|f(n + 1)$ thus $f(n) \neq 1$ and $f(n + 1) \neq 1$. This is a contradiction, so S can't have two distinct elements.

Solution 3 (Joseph Myers) (for the second part).

Claim 3.1. If $f(m) > 1$ for some $m \in \mathbb{N}$, then there are less than m different prime numbers p_i with $f(p_i) > 1$.

Proof. On the one hand, if $f(m) = c > 1$, then for any $n \in \mathbb{N}$, there is some $k \in \{n+1, n+2, \dots, n+m\}$ such that $f(k) \in \{1, c\}$. To show this, it is enough to see that condition (2) implies that for any $n \in \mathbb{N}$

- if $f(n) = c$, then $f(n+1) \in \{1, c\}$;
- if $f(n) = 1$, then $f(n+m) \in \{1, c\}$.

On the other hand, if we suppose that there are m different prime numbers p_1, p_2, \dots, p_m , then we know (by the Chinese Remainder Theorem) that for any number $t \in \mathbb{N}$, there exists a number $n_t \in \mathbb{N}$ such that $p_k^t \mid n_t + k$ for each $k \in \{1, 2, \dots, m\}$. Hence for each k , we have

$$f(n_t + k) \geq f(p_k^t) = f(p_k)^t \geq 2^t.$$

If $2^t > c$, then it means that $f(k) > c$ for all $k \in \{n_t + 1, n_t + 2, \dots, n_t + m\}$. \square

Now we can suppose that for some $m \geq 2$, the set of primes with $f(p) > 1$ is $S = \{p_1, p_2, \dots, p_m\}$. Now let S_1 and S_2 be two disjoint nonempty subsets such that $S_1 \cup S_2 = S$. Let $a = \prod_{p_i \in S_1} p_i$ and $b = \prod_{p_j \in S_2} p_j$.

Now for any $k, \ell \in \mathbb{N}$, $a^k + b^\ell$ is coprime to all primes in S , hence $f(a^k + b^\ell) = 1$. It is easy to see that we can choose k and ℓ such that $f(a^k) \neq f(b^\ell)$, which contradicts condition (2).

Solution 4 (for the main part). Let p the minimal (prime) number with $f(p) = c > 1$.

Claim 4.1. For any $k \in \mathbb{N}$, $f(1 + p + p^2 + \dots + p^k) \in \{1, c, c^2, \dots, c^k\}$.

Proof of the claim. Use condition (2) recursively. \square

Now suppose that there exists some prime $q > p$ with $f(q) > 1$, and consider the following three values of f :

- $f(1) = 1$.
- $f(p^{q-1} - 1) = f(p-1)f(p^{q-2} + p^{q-3} + \dots + p + 1) \in \{1, c, c^2, \dots, c^{q-2}\}$ using our claim, (note that we know $f(p-1) = 1$ by the minimality of p).
But we also know (by Fermat's little theorem) that $q \mid p^{q-1} - 1$, hence $f(q) \mid f(p^{q-1} - 1)$.
Therefore we have that: $f(p^{q-1} - 1) \in \{c, c^2, \dots, c^{q-2}\}$.
- $f(p^{q-1}) = c^{q-1}$.

These are three different values, hence we have a contradiction by condition (2).

Solution 4B (for the main part). Let p the minimal (prime) number with $f(p) = c > 1$.

Lemma 4B.1. For any $k \in \mathbb{N}$, $f(1 + p + p^2 + \dots + p^k) = 1$.

Proof of the Lemma, by induction. $k = 0$ is trivial.

$k = 1$ is also easy, as $p + 1$ cannot be a prime, except in the case of $p = 2$, which is also easy. (If $f(2) = c > 1$, then $f(3) \in \{1, c\}$ by $3 = 1 + 2$ and $f(3) \in \{1, c^2\}$ by $2^2 = 1 + 3$.)

From now we suppose $k \geq 2$ and $f(1 + p + \dots + p^{k-1}) = 1$. By condition (2), we have that:

- As $f(1 + p + \dots + p^{k-1}) = 1$ and $f(p^k) = c^k$,
we have that $f(1 + p + \dots + p^{k-1}) \in \{1, c^k\}$
- As $f(1) = 1$ and $f(p + \dots + p^k) = f(p)f(1 + p + \dots + p^{k-1}) = c$,
we also have that $f(1 + p + \dots + p^{k-1}) \in \{1, c\}$.

If $k > 1$, $c \neq c^k$, therefore $f(1 + p + \dots + p^k) = 1$. \square

Now suppose that there exists some prime $q > p$ with $f(q) > 1$. Then by condition (1),

$$f(p^{q-1} - 1) = f(p - 1)f(1 + p + \dots + p^{q-2})$$

here $f(p - 1) = 1$ (by the minimality of p) and $f(1 + p + \dots + p^{q-2}) = 1$ by the lemma. But we also know (by Fermat's little theorem) that $q \mid p^{q-1} - 1$, hence $1 < f(q) \leq f(p^{q-1} - 1)$, which is a contradiction.

Solution 5 (for the main part).

Lemma 5.1. If $f(m) \neq f(n)$, then $f(n + m) = f(n - m)$ (for all $n > m \in \mathbb{N}$).

Proof of the Lemma. By condition (2), $\{f(n + m), f(n - m)\} \subseteq \{f(n), f(m)\}$. We also have that:

$$f(n - m)f(n + m) = f(n^2 - m^2) \in \{f(n)^2, f(m)^2\}$$

which is only possible if $f(n - m) = f(n + m) = f(n)$ or $f(n - m) = f(n + m) = f(m)$. \square

Now let p be the smallest (prime) number with $f(p) > 1$.

Claim 5.2. If p does not divide n , then $f(n) = 1$.

Proof of the Claim. Let n be the minimal counterexample, and let $n = dp + r$ with $0 < r < p$. Here $d \geq 1$ by the minimality of p .

Now $1 < f(d)f(p) = f(dp) \neq f(r) = 1$, hence by our lemma, $f(dp + r) = f(dp - r) = 1$, contradiction.

Problem 3.

An infinite sequence of positive integers a_1, a_2, \dots is called *good* if

- (1) a_1 is a perfect square, and
- (2) for any integer $n \geq 2$, a_n is the smallest positive integer such that

$$na_1 + (n-1)a_2 + \dots + 2a_{n-1} + a_n$$

is a perfect square.

Prove that for any good sequence a_1, a_2, \dots , there exists a positive integer k such that $a_n = a_k$ for all integers $n \geq k$.

Proposed by: Joe Benton and Dominic Yeo, United Kingdom

Solution. Define the following auxiliary sequences:

$$b_1 = a_1, \quad b_n = a_1 + a_2 + \dots + a_n,$$

$$c_1 = b_1, \quad c_n = b_1 + b_2 + \dots + b_n.$$

Observe that

$$c_n - c_{n-1} = b_n,$$

$$b_n - b_{n-1} = (c_n - c_{n-1}) - (c_{n-1} - c_{n-2}) = a_n.$$

Therefore,

$$c_n = (2c_{n-1} - c_{n-2}) + a_n$$

is the smallest square greater than $2c_{n-1} - c_{n-2}$.

Claim. $\sqrt{c_n} - \sqrt{c_{n-1}} \leq \sqrt{c_{n-1}} - \sqrt{c_{n-2}}$

Proof 1. Let $c_n = m_n^2$, $c_{n-1} = m_{n-1}^2$, $c_{n-2} = m_{n-2}^2$, where the value of c_n is momentarily permitted to exceed its minimum. Any value is permitted for which

$$b_n = c_n - c_{n-1} > b_{n-1} = c_{n-1} - c_{n-2}.$$

Factoring,

$$(m_n - m_{n-1})(m_n + m_{n-1}) > (m_{n-1} - m_{n-2})(m_{n-1} + m_{n-2}).$$

In the case that $m_n - m_{n-1} = m_{n-1} - m_{n-2}$, this inequality clearly holds, as $m_n + m_{n-1} > m_{n-1} + m_{n-2}$.

Thus, the minimal possible value of m_n satisfies the claimed inequality.

Proof 2. Denote $c_{n-1} = x^2$, $c_{n-2} = (x-d)^2$. Then

$$2c_{n-1} - c_{n-2} = 2x^2 - (x-d)^2 = x^2 + 2dx - d^2 = (x+d)^2 - 2d^2 < (x+d)^2.$$

It follows that $c_n \leq (x+d)^2$. ■

And so the sequence of positive integers $\sqrt{c_n} - \sqrt{c_{n-1}}$ is decreasing. Any such sequence is eventually constant.

As a corollary, $c_n = (x+nd)^2$ for sufficiently large n , with fixed integers x and d . Along the lines above, it becomes clear that $a_n = c_n - 2c_{n-1} + c_{n-2} = 2d^2$, so the sequence (a_n) is constant.

Solution. We write:

$$s_n^2 = S_n = a_1 + (a_1 + a_2) + \dots + (a_1 + \dots + a_n).$$

So, setting $b_n := a_1 + \dots + a_n$, we have $S_n = b_1 + b_2 + \dots + b_n$ and, in particular $S_{n+1} = S_n + b_{n+1}$. Now, we study the quantity $S_n b_n = b_1 + b_2 + \dots + b_n + b_n$ in two different ways. Since b_{n+1} is the smallest integer strictly greater than b_n such that $b_1 + \dots + b_n + b_{n+1}$ is a perfect square, we must have

$$b_1 + b_2 + \dots + b_n + b_n \geq (s_{n+1} - 1)^2.$$

However, we also have

$$b_1 + b_2 + \dots + b_n + b_n = S_n + b_n = 2S_n - S_{n-1}.$$

Combining, we obtain

$$s_n^2 \geq \frac{s_{n-1}^2 + (s_{n+1} - 1)^2}{2} > \left(\frac{s_{n-1} + s_{n+1} - 1}{2} \right)^2.$$

where the final inequality is strict since the sequence (s_k) is strictly increasing. Taking a square root, and noting that all the (s_n) are integers, one obtains $s_{n+1} - s_n \leq s_n - s_{n-1}$.

Now we focus on the sequence $d_n = s_{n+1} - s_n$. (s_k) is strictly increasing thus (d_k) is positive. However we proved that $d_{n+1} \leq d_n$, so the sequence (d_k) is eventually constant, so eventually $s_n = bn + c$ and $S_n = (bn + c)^2$ with some numbers b, c ; then

$$a_{n+2} = S_{n+2} - 2S_{n+1} + S_n = (b(n+2) + c)^2 - 2(b(n+1) + c)^2 + (bn + c)^2 = 2b^2.$$

Remark. The key idea is to use the D difference operator on the space of sequences. That is a function which takes a sequence as an input and outputs another sequence the following way: if (x) is the input sequence, then

$$D(x) = D((x_1, x_2, x_3, x_4, \dots)) = (x_2 - x_1, x_3 - x_2, x_4 - x_3, \dots).$$

Here $DD(S) = (a)$ (with some shift), $D(s) = (d)$. This D operator has some nice properties; if someone studied calculus, these properties will be familiar. $D(x + y) = D(x) + D(y)$; $D(\lambda x) = \lambda D(x)$; (x) constant iff Dx zero; (x) is linear iff Dx constant; in general (x) is a polynomial with degree $l + 1$ iff $D(x)$ is a polynomial with degree l .

First we proved that sequence (d) is eventually constant, thus $D(d) = DD(s)$ is eventually zero. Therefore the sequence (s) is eventually a linear function, hence $s^2 = S$ is eventually a quadratic polynomial, therefore $0 = DDD(S) = D(a)$, so (a) is constant eventually.

Problem 4.

Given a positive integer $n \geq 2$, determine the largest positive integer N for which there exist $N + 1$ real numbers a_0, a_1, \dots, a_N such that

- (1) $a_0 + a_1 = -\frac{1}{n}$, and
- (2) $(a_k + a_{k-1})(a_k + a_{k+1}) = a_{k-1} - a_{k+1}$ for $1 \leq k \leq N - 1$.

Proposed by: Romania

Solution 1. $(a_k + a_{k-1})(a_k + a_{k+1}) = a_{k-1} - a_{k+1}$ is equivalent to $(a_k + a_{k-1} + 1)(a_k + a_{k+1} - 1) = -1$. Let $b_k = a_k + a_{k+1}$. Thus we need b_0, b_1, \dots the following way: $b_0 = -\frac{1}{n}$ and $(b_{k-1} + 1)(b_k - 1) = -1$. There is a proper sequence b_0, b_1, \dots, b_{N-1} if and only if there is proper sequence a_0, a_1, \dots, a_N , because from a proper (a_k) sequence we can get a proper (b_k) sequence with $b_k = a_k + a_{k+1}$ for $k = 0, 1, \dots, N - 1$ and from a proper (b_k) sequence we can get a proper (a_k) sequence by arbitrarily setting a_0 and then inductively defining $a_k = b_{k-1} - a_{k-1}$ for $k = 1, 2, \dots, N$.

We prove by induction that $b_k = -\frac{1}{n-k}$ for $k < n$. This is true for $k = 0$, as $b_0 = -\frac{1}{n}$ and

$$b_k = 1 - \frac{1}{b_{k-1} + 1} = 1 - \frac{1}{1 - \frac{1}{n-k+1}} = -\frac{1}{n-k}$$

for $k < n$. Thus there is a proper sequence b_0, b_1, \dots, b_{n-1} , but it can't be continued, because $b_{n-1} + 1 = 0$ so there is no b_n for which $(b_{n-1} + 1)(b_n - 1) = -1$.

Therefore the longest proper sequence (b_k) is n -long, so the longest proper sequence (a_k) is $n + 1$ long, so $N = n$.

Solution 2. The required maximum is $N = n$.

To rule out the case $N \geq n + 1$, it is clearly sufficient to rule out the case $N = n + 1$.

Assume for contradiction that a_0, a_1, \dots, a_{n+1} are real numbers satisfying both conditions in the statement. It is sufficient to show that $a_k + a_{k+1} = 0$ for some $k \leq n$, because then $a_{k-1} - a_{k+1} = 0$ so $a_{k+1} = a_{k-1}$, therefore $a_{k-1} + a_k = 0$ and so on, by backwards recursion we get that $a_j + a_{j+1} = 0$ for all $0 \leq j \leq k$, but this is a contradiction with $a_0 + a_1 = -\frac{1}{n}$.

To prove that $a_k + a_{k+1} = 0$ for some $k \leq n$, assume that $a_k + a_{k+1} \neq 0$ for all $k \leq n$, to rewrite the second condition in the statement in the form

$$\frac{1}{a_k + a_{k+1}} - \frac{1}{a_{k-1} + a_k} = 1, \quad k = 1, \dots, n.$$

and sum both sides over the full range from $k = 1$ to n . This gives

$$\frac{1}{a_n + a_{n+1}} - \frac{1}{a_0 + a_1} = n.$$

As $a_0 + a_1 = -\frac{1}{n}$, this means that $\frac{1}{a_n + a_{n+1}} = 0$, which is a contradiction. Consequently, $N \leq n$.

To provide $n + 1$ real numbers satisfying both conditions in the statement, fix a_0 and go through the telescoping procedure above to obtain

$$a_k = (-1)^k a_0 + \sum_{j=1}^k \frac{(-1)^{k-j+1}}{n-j+1}, \quad k = 1, \dots, n.$$

This concludes the proof.

Problem 5.

For all positive integers n, k , let $f(n, 2k)$ be the number of ways an $n \times 2k$ board can be fully covered by nk dominoes of size 2×1 . (For example, $f(2, 2) = 2$ and $f(3, 2) = 3$.)

Find all positive integers n such that for every positive integer k , the number $f(n, 2k)$ is odd.

Proposed by: U.S.A.

Solution. The integers n with the requested property are exactly the ones of the form $2^k - 1$.

In what follows, let $f(m, n)$ denote the number of domino tilings of an $m \times n$ grid. (For convenience, we also allow m and n to be 0, in which case $f(m, n) = 1$.)

Claim. $f(m, 2n + 1) \equiv f(m, n) \pmod{2}$ for all n and even m

Proof. Consider reflecting $m \times (2n + 1)$ tilings across the central column. In this way, the tilings are grouped into singletons and pairs, so modulo 2, $f(m, 2n + 1)$ is congruent to the number of singletons.

If a tiling is invariant under reflection in the central column, then every domino with one cell in that column must have both cells in that column.

In other words, the central column is filled with $\frac{m}{2}$ vertical dominoes, splitting the remainder of the grid into two $m \times n$ grids. Obeying the symmetry constraint, there are $f(m, n)$ tilings: each of the tilings of the left grid fixes the tiling of the right grid.

Claim. $f(n, n) \equiv 0 \pmod{2}$ for all even $n \geq 2$. (Recall that $f(0, 0) = 1$ is odd.)

Proof. Consider reflecting $n \times n$ tilings across the diagonal. This groups the tilings into pairs, (no tiling is grouped with itself). Hence the number of $n \times n$ tilings is even.

We are ready to complete the solution.

- If n is odd, the first claim shows that n satisfies the property if and only if $\frac{1}{2}(n - 1)$ does.
- If $n \geq 2$ is even, the second claim shows that n does not satisfy the property.
- If $n = 0$, then n satisfies the property, as $f(m, 0) = 1$ always.

This concludes the proof that the sought numbers are the ones of the form $2^k - 1$.

Solution 2.

Color the board as a chessboard. Consider the bipartite graph whose vertices are the squares and the neighbors are connected by an edge. Notice that a domino tiling described in the problem corresponds to a perfect matching in this bipartite graph, so we are interested in the number of perfect matchings.

And that is the permanent of the (bipartite) adjacency matrix. However we need to compute it mod 2 (over F_2), so that is the determinant of that matrix.

So our goal is to decide whether the determinant is 0 or 1 mod 2, or in other words, that matrix is singular or not. From now we compute everything mod 2. Now we understand what does it mean, that a vector v is in the nullspace (has eigenvalue 0). But each entry of v is 0 – 1, so in other words we choose a subset of the black squares (v is the characteristic vector of that subset). So in the language of the board, there is a nontrivial v in the nullspace iff we have a subspace of black square such a way that each white square has even number of black chosen neighbors.

If n is even, then we can choose a subset such a way in the $n \times n$ square: the diagonal.

Now we prove similar connection between n and $2n + 1$: there is a construction for n iff we have one for $2n + 1$. k is always even. If we have a construction in $n \times k$ then we have one in $(2n + 1) \times k$: we mirror the construction to the middle row.

Now we prove that from a construction in $(2n + 1) \times k$ we can make a construction in $n \times k$. First, we

don't have any chosen black square in the middle row. Then the chosen black squares in the upper (or lower) $n \times k$ rectangle is a construction for $n \times k$. Second, we have some chosen black square in the middle row. Because we can find a white square in the middle row such that it has only one chosen black neighbor in the middle row. Thus this construction isn't symmetric to the horizontal line the symmetric difference of the construction and its reflection (to the vertical line) is a nontrivial vector in the nullspace, thus we can use the first case.

Problem 6.

Let $ABCD$ be a cyclic quadrilateral with circumcentre O . Let the internal angle bisectors at A and B meet at X , the internal angle bisectors at B and C meet at Y , the internal angle bisectors at C and D meet at Z , and the internal angle bisectors at D and A meet at W . Further, let AC and BD meet at P . Suppose that the points X, Y, Z, W, O and P are distinct.

Prove that O, X, Y, Z and W lie on the same circle if and only if P, X, Y, Z and W lie on the same circle.

Proposed by: Ethan Tan, Australia

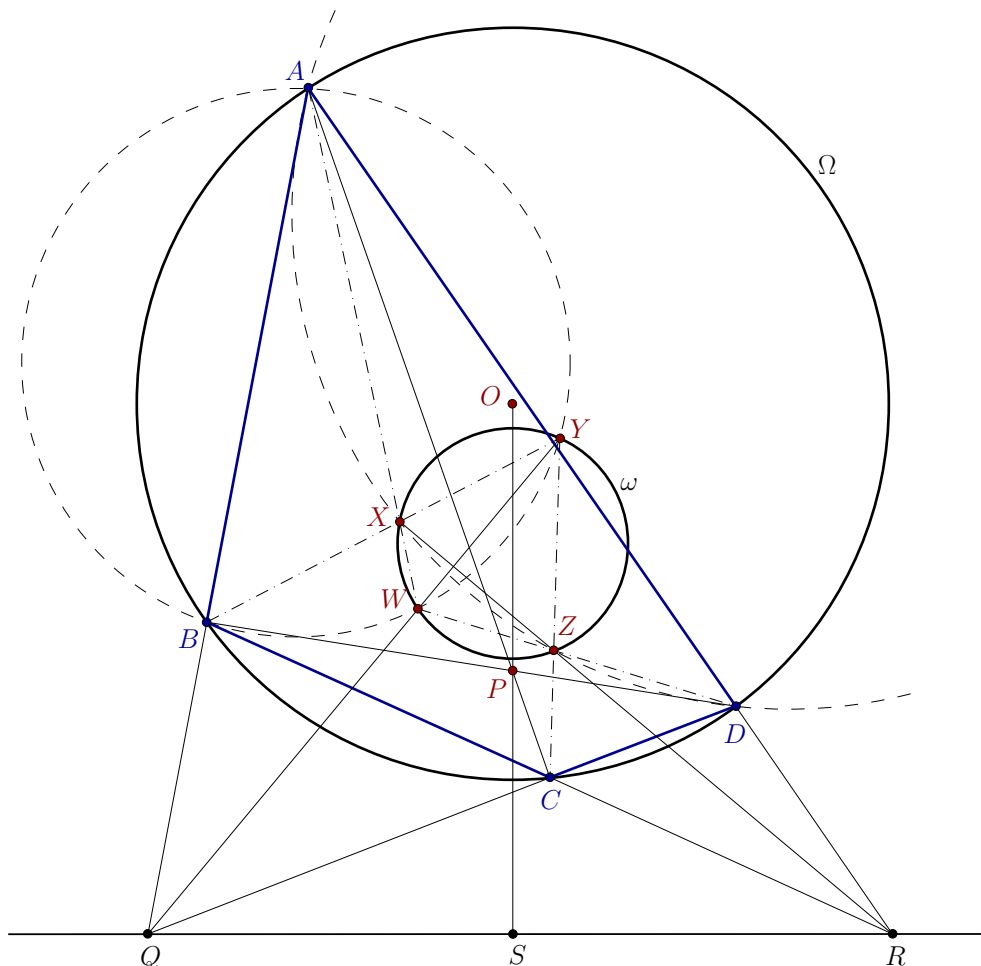
Solution 1. Let Ω be the circumcircle of the quadrilateral $ABCD$ and let r be its radius.

First, notice that the points X, Y, Z, W are concyclic. Indeed, using oriented (modulo 180°) angles,

$$\angle(XW, XY) + \angle(ZY, ZW) = \angle(XA, XB) + \angle(ZC, ZD) = -\frac{\angle A + \angle B}{2} - \frac{\angle C + \angle D}{2} = 0.$$

Let ω be the circle passing through these four points. Our goal is to prove that $O \in \omega$ occurs if and only if $P \in \omega$.

Next, we rule out the case when $ABCD$ is a trapezium. Suppose that if $ABCD$ is an isosceles trapezium; without loss of generality say $AB \parallel CD$. By symmetry, points X, Z, O and P lie on the symmetry axis (that is, the common perpendicular bisector of AB and CD), therefore they are collinear. By the conditions of the problem, these four points are distinct and X and Z are on ω , so neither O , nor P can lie on ω ; therefore the problem statement is obvious. From now on we assume that $AB \not\parallel CD$ and $BC \not\parallel AD$.



Let AB and CD meet at Q , and let BC and AD meet at R . Without loss of generality, suppose that B lies between A and Q , and D lies between A and R . Now we show that line QR is the radical axis between circles Ω and ω .

Point W is the intersection point of the bisectors of $\angle A$ and $\angle D$, so in triangle ADQ , point W is the incentre. Similarly, BY and CY are the external bisectors of $\angle QBC$ and $\angle BCQ$, so in triangle BCQ , point Y is the excentre opposite to Q . Hence, both Y and W lie on the bisector of $\angle DQA$.

By $\angle DQA = 180^\circ - \angle D - \angle A = \angle B - \angle A$ we get

$$\angle BYQ = \angle YBA - \angle YQB = \frac{\angle B}{2} - \frac{\angle DQA}{2} = \frac{\angle B}{2} - \frac{\angle B - \angle A}{2} = \frac{\angle A}{2} = \angle BAW,$$

so the points A, B, Y, W are concyclic, and therefore $QA \cdot QB = QY \cdot QW$; hence, Q has equal power with respect to Ω and ω . It can be seen similarly that R has equal power with respect to the circles Ω and ω . Hence, the radical axis of Ω and ω is line QR .

Let lines OP and QR meet at point S . It is well-known that with respect to circle $ABCD$, the diagonal triangle PQR is autopolar. As consequences we have $OP \perp QR$, and the points P and S are symmetric to circle $ABCD$, so $OS \cdot OP = r^2$.

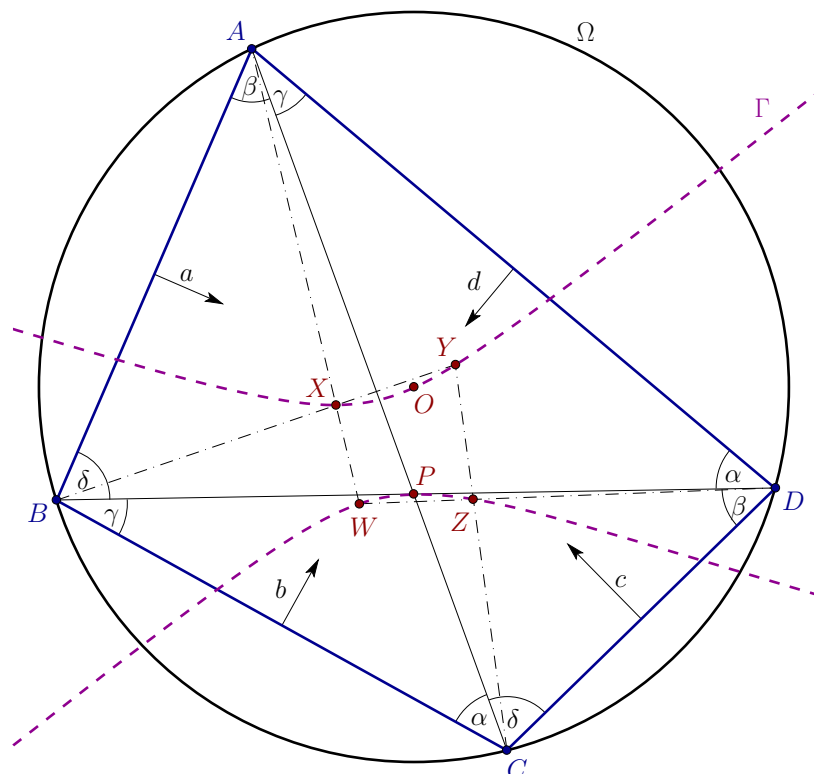
Notice that P and O lie inside Ω , so the polar line QR lies entirely outside, so S is different from P and O . Moreover,

$$SO \cdot SP = OS \cdot (OS - OP) = OS^2 - OS \cdot OP = OS^2 - r^2,$$

so $SO \cdot SP$ is equal to the power of S with respect to Ω . Since S lies on the radical axis QR , it has equal power with respect to the two circles; therefore, $SO \cdot SP$ is equal to the power of S with respect to ω . From this, it follows that $O \in \omega$ if and only if $P \in \omega$.

Solution 2. We will prove that the points X, Y, Z, W, O, P lie on a conic section Γ . Five distinct points uniquely determine a conic section, therefore

$$X, Y, Z, W, O \text{ are concyclic} \Leftrightarrow \Gamma \text{ is a circle} \Leftrightarrow X, Y, Z, W, P \text{ are concyclic.}$$



Let $\alpha = \angle ACB = \angle ADB$, $\beta = \angle BDC = \angle BAC$, $\gamma = \angle CAD = \angle CBD$ and $\delta = \angle DBA = \angle DCA$, where of course $\alpha + \beta + \gamma + \delta = 180^\circ$. For every point ξ in the plane, let $a(\xi)$, $b(\xi)$, $c(\xi)$ and $d(\xi)$ be the signed distances between ξ and the lines AB , BC , CD and DA , respectively, such that the quadrilateral lies on the positive sides of these lines.

The equations of the bisectors of $\angle AXW$, $\angle BXY$, $\angle CYZ$ and $\angle DZW$ are $a(\xi) - d(\xi) = 0$, $b(\xi) - a(\xi) = 0$, $c(\xi) - b(\xi) = 0$ and $d(\xi) - c(\xi) = 0$, respectively. Notice that the points X, Y, Z, W satisfy the quadratic equations $(a(\xi) - d(\xi))(b(\xi) - c(\xi)) = 0$ and $(a(\xi) - b(\xi))(c(\xi) - d(\xi)) = 0$, so

$$a(\xi)b(\xi) + c(\xi)d(\xi) = a(\xi)c(\xi) + b(\xi)d(\xi) = a(\xi)d(\xi) + b(\xi)c(\xi) \quad \text{for } \xi = X, Y, Z, W. \quad (1)$$

Without loss of generality, suppose that $ABCD$ is inscribed in a unit circle. Then

$$a(O) = \cos \alpha, \quad b(O) = \cos \beta, \quad c(O) = \cos \gamma, \quad d(O) = \cos \delta. \quad (2a)$$

By $\frac{a(P)}{b(P)} = \frac{BP \cdot \sin \delta}{BP \cdot \sin \gamma} = \frac{1/\sin \gamma}{1/\sin \delta}$ and the analogous relations we can see that

$$a(P) = \frac{k}{\sin \gamma}, \quad b(P) = \frac{k}{\sin \delta}, \quad c(P) = \frac{k}{\sin \alpha}, \quad d(P) = \frac{k}{\sin \beta} \quad (2b)$$

with some positive number k .

Now let $s = \frac{\sin \alpha \sin \beta \sin \gamma \sin \delta}{k^2}$; then, by (2a) and (2b),

$$\begin{aligned} a(O)b(O) - s \cdot a(P)b(P) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta = \cos(\alpha + \beta) \\ c(O)d(O) - s \cdot c(P)d(P) &= \cos(\gamma + \delta) = -\cos(\alpha + \beta) \\ a(O)b(O) + c(O)d(O) &= s \cdot (a(P)b(P) + c(P)d(P)); \end{aligned} \quad (3a)$$

analogous calculation provides

$$a(O)c(O) + b(O)d(O) = s \cdot (a(P)c(P) + b(P)d(P)) \quad (3b)$$

$$a(O)d(O) + b(O)c(O) = s \cdot (a(P)d(P) + b(P)c(P)). \quad (3c)$$

In order to find the equation of the curve Γ , choose real numbers u, v, w , not all zero, such that

$$u + v + w = 0; \quad (4)$$

$$u \cdot (a(P)b(P) + c(P)d(P)) + v \cdot (a(P)c(P) + b(P)d(P)) + w \cdot (a(P)d(P) + b(P)c(P)) = 0. \quad (5)$$

This is always possible, because this a system of two homogeneous linear equations with three variables. Then the equation of Γ will be

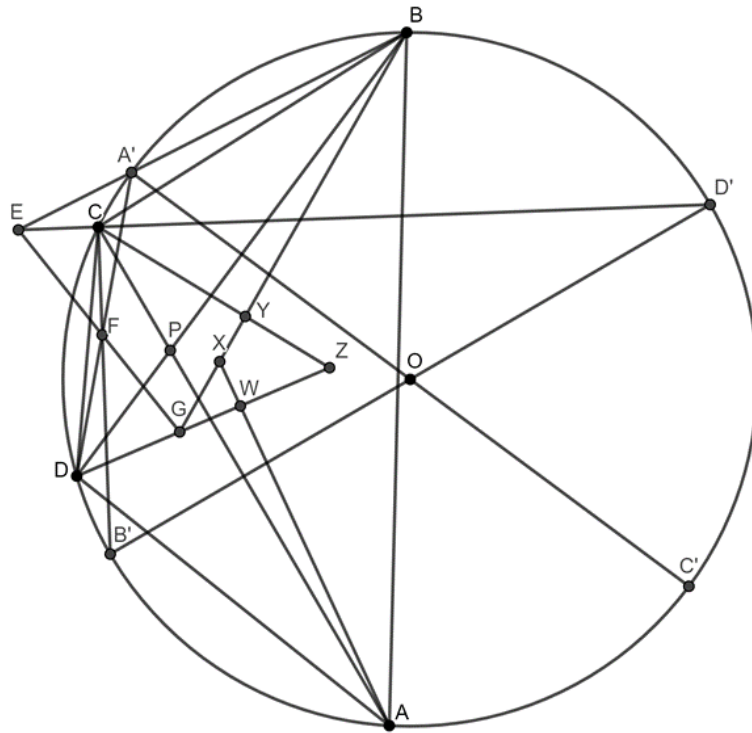
$$f(\xi) = u \cdot (a(\xi)b(\xi) + c(\xi)d(\xi)) + v \cdot (a(\xi)c(\xi) + b(\xi)d(\xi)) + w \cdot (a(\xi)d(\xi) + b(\xi)c(\xi)) = 0. \quad (*)$$

As can be seen, $f(X) = f(Y) = f(Z) = f(W) = 0$ follows from (1) and (4), $f(P) = 0$ follows from (5), then $f(O) = s \cdot f(P) = 0$ follows from (3a – 3c). So, the points X, Y, Z, W, O, P all satisfy the equation $f(\xi) = 0$.

Notice that $f(B) = u \cdot c(B)d(B)$ and $f(A) = w \cdot b(A)c(A)$; since at least one of u and w is nonzero, either A or B does not satisfy (*). Therefore, the equation cannot degenerate to an identity.

Hence, the equation $f(\xi) = 0$ is at most quadratic, it is not an identity, but satisfied by X, Y, Z, W, O, P , so these six points lie on a conic section.

Solution 3. Let A' , B' , C' and D' be the second intersection of the angle bisectors of $\angle DAC$, $\angle ABC$, $\angle BCD$ and $\angle CDA$ with the circumcircle of $ABCD$, respectively. Notice that A' and C' are the midpoints of the two arcs of BD and B' and D' are the midpoints of arcs corresponding to AC so the lines $A'C'$ and $B'D'$ intersect in O .



We will prove that the points X, Y, Z, W, P and O lie on a conic section. This is a more general statement because five points uniquely determine a conic sections so if X, Y, Z, W and P lie on a circle then this circle is the conic section so O also lies on this circle. In the same way if X, Y, Z, W and O are on a circle then it also contains P .

Let E be the intersection of the lines $A'B$ and CD' . Let F be the intersection of the lines $A'D$ and $B'C$. Finally, let G be the intesection of XY and ZW .

Now we use Pascal's theorem five times:

1. From $BDD'CAA'$ we see that the points P, W and E are collinear.
2. From $BB'D'CC'A'$ we see that the points Y, O and E are collinear.
3. From $B'BDA'AC$ we see that the points X, P and F are collinear.
4. From $B'D'DA'C'C$ we see that the points O, Z and F are collinear.
5. From $D'CB'BA'D$ we see that the points E, F and G are collinear.

So the opposite sides of the hexagon $PXYOZW$ intersect at points F, G and E , which are collinear, so by the converse of Pascal's theorem the points X, Y, Z, W, P and O indeed lie on a conic section.