Solutions of EGMO 2020

Problem 1. The positive integers a_0, a_1, a_2, …, a_{3030} satisfy

$$2a_{n+2} = a_{n+1} + 4a_n \quad \text{for } n = 0, 1, 2, \ldots, 3028.$$

Prove that at least one of the numbers a_0, a_1, a_2, …, a_{3030} is divisible by 2^{2020}.

Problem 2. Find all lists $(x_1, x_2, \ldots, x_{2020})$ of non-negative real numbers such that the following three conditions are all satisfied:

(i) $x_1 \le x_2 \le \ldots \le x_{2020}$;

(ii) $x_{2020} \le x_1 + 1$;

(iii) there is a permutation $(y_1, y_2, \ldots, y_{2020})$ of $(x_1, x_2, \ldots, x_{2020})$ such that

$$\sum_{i=1}^{2020} \left((x_i + 1)(y_i + 1)\right)^2 = 8 \sum_{i=1}^{2020} x_i^3.$$

A permutation of a list is a list of the same length, with the same entries, but the entries are allowed to be in any order. For example, $(2, 1, 2)$ *is a permutation of* $(1, 2, 2)$*, and they are both permutations of* $(2, 2, 1)$*. Note that any list is a permutation of itself.*

Problem 3. Let $ABCDEF$ be a convex hexagon such that $\angle A = \angle C = \angle E$ and $\angle B = \angle D = \angle F$ and the (interior) angle bisectors of $\angle A$, $\angle C$, and $\angle E$ are concurrent.

Prove that the (interior) angle bisectors of $\angle B$, $\angle D$, and $\angle F$ must also be concurrent.

Note that $\angle A = \angle FAB$*. The other interior angles of the hexagon are similarly described.*

Problem 4. A permutation of the integers 1, 2, …, m is called *fresh* if there exists no positive integer $k < m$ such that the first k numbers in the permutation are 1, 2, …, k in some order. Let f_m be the number of fresh permutations of the integers 1, 2, …, m.

Prove that $f_n \ge n \cdot f_{n-1}$ for all $n \ge 3$.

For example, if $m = 4$*, then the permutation* $(3, 1, 4, 2)$ *is fresh, whereas the permutation* $(2, 3, 1, 4)$ *is not.*

Problem 5. Consider the triangle ABC with $\angle BCA > 90°$. The circumcircle Γ of ABC has radius R. There is a point P in the interior of the line segment AB such that $PB = PC$ and the length of PA is R. The perpendicular bisector of PB intersects Γ at the points D and E.

Prove that P is the incentre of triangle CDE.

Problem 6. Let $m > 1$ be an integer. A sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = a_2 = 1$, $a_3 = 4$, and for all $n \ge 4$,

$$a_n = m(a_{n-1} + a_{n-2}) - a_{n-3}.$$

Determine all integers m such that every term of the sequence is a square.

Solutions to Problem 1

There are different ways of solving the problem. All of these use some induction argument. Most of these proofs use one of the following two lemmas. In many places, they can be used interchangeably.

Lemma. If a, b, c, d are integers with $2c = b + 4a$ and $2d = c + 4b$, then $4 \mid b$.

Proof: From $2d = c + 4b$ we have that c is even, and then from $2c = b + 4a$ it follows that b is divisible by 4.

Lemma'. For $0 \le n \le 3030$, denote by v_n the largest integer such that 2^{v_n} divides a_n. We claim the following:

$$(*) \qquad v_{n+1} \ge \min(v_n + 2, v_{n+2} + 1) \qquad \text{for } n = 0, 1, \ldots, 3028.$$

Proof: Let $0 \le n \le 3028$ and let $s = \min(v_n + 2, v_{n+2} + 1)$. Then $s \le v_n + 2$ implies $2^s \mid 4a_n$ and $s \le v_{n+2} + 1$ implies $2^s \mid 2a_{n+2}$. It follows that $a_{n+1} = 2a_{n+2} - 4a_n$ is also divisible by 2^s, hence $s \le v_{n+1}$, which proves $(*)$.

Here are different ways of working out the induction argument that is crucial in the proofs.

Induction part, alternative A.

Statement: For $k = 0, 1, \ldots, 1010$, the terms $a_k, a_{k+1}, \ldots, a_{3030-2k}$ are all divisible by 2^{2k}.

Reformulation of the statement using notation v_n is the largest integer such that 2^{v_n} divides a_n: we have $v_n \ge k$ for any n satisfying $\lceil \frac{1}{2}k \rceil \le n \le 3030 - k$. Here $\lceil x \rceil$ denotes the smallest integer not smaller than x.

Proof 1: We proceed by induction on k. For $k = 0$ the statement is obvious, so, for the inductive step, suppose that $a_k, a_{k+1}, \ldots, a_{3030-2k}$ are all divisible by 2^{2k}. Apply the Lemma with

$$(a, b, c, d) = \left(\frac{a_{i-1}}{2^{2k}}, \frac{a_i}{2^{2k}}, \frac{a_{i+1}}{2^{2k}}, \frac{a_{i+2}}{2^{2k}}\right)$$

for $i = k+1, k+2, \ldots, 3030 - 2k - 2$. We obtain that $\frac{a_i}{2^{2k}}$ is divisible by 4 (and hence a_i is divisible by 2^{2k+2}) for $i = k+1, k+2, \ldots, i = 3030 - 2k - 2$. This completes the induction. For $k = 1010$ we obtain that a_{1010} is divisible by 2^{2020}, and the solution is complete. □

Remark. Remark, notice that by replacing 1010 with n (and hence, $2n$ with 2020 and $3n$ with 3030, this argument works, too. Then the claim is the following: if $a_0, a_1, \ldots, a_{3n}$ are integers that satisfy the recursion in the problem, then a_n is divisible by 2^{2n}.

Proof 2: We will show by two step induction to $k \ge 0$: we have $v_n \ge k$ for any n satisfying $\lceil \frac{1}{2}k \rceil \le n \le 3030 - k$. Here $\lceil x \rceil$ denotes the smallest integer not smaller than x. Plugging in $k = 2020$ and $n = 1010$ will give the desired result.
The case $k = 0$ is trivial, since the v_n are non-negative. For the case $k = 1$, let $1 \le n \le 3029$, then $v_n \ge \min(v_{n-1} + 2, v_{n+1} + 1) \ge 1$.
Suppose we have it proven for some $k \ge 1$ and for $k - 1$. Let $\lceil \frac{1}{2}(k+1) \rceil \le n \le 3030 - (k+1)$. Then $\lceil \frac{1}{2}(k-1) \rceil \le n - 1 \le 3030 - (k-1)$ and also $\lceil \frac{1}{2}k \rceil \le n + 1 \le 3030 - k$. By induction hypothesis we have $v_{n-1} \ge k - 1$ and $v_{n+1} \ge k$. Then $v_n \ge \min(v_{n-1} + 2, v_{n+1} + 1) \ge k$, finishing the induction. □

Proof 3: The notation is the same as in alternative A1, but the induction step has two steps. We use the lemma stated in the beginning as the first step. Assume now that $2^{2k} \mid a_k, \ldots, a_{3030-2k}$ for some $k \ge 1$. We claim that then $2^{2k+2} \mid a_{k+1}, \ldots, a_{3030-2(k+1)}$. Notice first that any i on the interval $[k+1, 3030 - 2k - 1]$ satisfies the equation $2a_{i+1} = a_i + 4a_{i-2}$, we have $2^{2k+1} \mid a_i$. Furthermore, if $i \in [k+1, 3030 - 2k - 2]$, we have $2^{2k+2} \mid a_i$ since $2^{2k+1} \mid a_{i+1}$ and hence

$2^{2k+2} \mid 2a_{i+1}$ and $2^{2k+2} \mid 4a_{i-1}$. $\square$

Induction part, alternative B.

Statement: if a_0, a_1, …, a_{3n} are integers that satisfy the recursion in the problem, then a_n is divisible by 2^{2n}. The problem statement follows for $n = 1010$.

Proof: The base case of the induction is exactly the Lemma we just proved. Now, for the inductive step, suppose that the statement holds for $n = k-1$, and consider integers a_0, a_1, …, a_{3k} that satisfy the recursion. By applying the induction hypothesis to the four sequences $(a_0, a_1, \ldots, a_{3k-3})$, $(a_1, a_2, \ldots, a_{3k-2})$, $(a_2, a_3, \ldots, a_{3k-1})$ and $(a_3, a_4, \ldots, a_{3k})$, we find that a_{k-1}, a_k, a_{k+1} and a_{k+2} are all divisible by 2^{2k-2}. If we now apply the lemma to $a_{k-1}/2^{2k-2}$, $a_k/2^{2k-2}$, $a_{k+1}/2^{2k-2}$ and $a_{k+2}/2^{2k-2}$, we find that $a_k/2^{2k-2}$ is divisible by 4, so a_k is divisible by 2^{2k}, as desired. $\square$

Induction part, alternative C.

Statement: Given positive integers a_0, a_1, a_2, …, a_{3k} such that

$$2a_{n+2} = a_{n+1} + 4a_n \quad \text{for } n = 0, 1, 2, \ldots, 3k-2,$$

then 2^{2k} divides at least one of the numbers a_0, a_1, a_2, …, a_{3k}.

Proof: The case $k = 1$ is obtained from Lemma.

Suppose that for some $k \geq 1$, our claim is true for any sequence of $3k+1$ positive integers that satisfy similar defining relations. Then consider a sequence of $3(k+1)+1 = 3k+4$ positive integers a_0, a_1, a_2, …, a_{3k+3} such that

$$2a_{n+2} = a_{n+1} + 4a_n \quad \text{for } n = 0, 1, 2, \ldots, 3k+1.$$

Then, we see that all numbers a_1, a_2,…, a_{3k+2} are even, and so we may set $a_i = 2b_i$ for some positive integers b_i for $i = 1, 2, \ldots, 3k+2$. Then $2a_2 = a_1 + 4a_0$ becomes

$$2b_2 = b_1 + 2a_0.$$

Note that

$$2b_{n+2} = b_{n+1} + 4b_n \quad \text{for } n = 1, 2, \ldots, 3k.$$

Hence the positive integers b_1, b_2, …, b_{3k+1} are even, and so we may write $b_i = 2c_i$ for some positive integers c_i for $i = 1, 2, \ldots, 3k+1$. Then $2b_2 = b_1 + 2a_0$ becomes $2c_2 = c_1 + a_0$, which does not give us anything. However, we have

$$2c_{n+2} = c_{n+1} + 4c_n \quad \text{for } n = 1, 2, \ldots, 3k-1.$$

So the sequence of positive integers c_1, c_2, …, c_{3k+1} is a sequence of $3k+1$ positive integers that satisfy the defining relations. By the inductive hypothesis, 2^{2k} divides at least one of the numbers c_1, c_2, …, c_{3k+1}. Since $a_i = 4c_i$ for all $i = 1, 2, \ldots, 3k+1$, it follows that $4 \cdot 2^{2k} = 2^{2(k+1)}$ divides at least one of the numbers $a_1, \ldots, a_{3k+1}$. This completes the induction, and hence the proof. $\square$

Induction part, alternative D.

Statement: If v_0, v_1, …, v_{3030} is a sequence of non-negative integers satisfying $(*)$, there must be a k such that $v_k \geq 2020$. In fact, we will show that $v_{1010} \geq 2020$.

These proofs use Lemma'.

Proof 1: We will show by induction on k that $v_n \geq 2k$ for $k \leq n \leq 3030-2k$ and $v_n \geq 2k+1$ for $k+1 \leq n \leq 3030-2k-1$. For $k=1010$, the first statement implies that $v_{1010} \geq 2020$.
For $k=0$ the first statement $v_n \geq 0$ is obvious, and the second statement follows using $(*)$: we have $v_n \geq v_{n+1}+1 \geq 1$ for $1 \leq n \leq 3029$.
Now suppose that the inductive hypothesis holds for $k=\ell$: we have $v_n \geq 2\ell$ for $\ell \leq n \leq 3030-2\ell$ and $v_n \geq 2\ell+1$ for $\ell+1 \leq n \leq 3030-2\ell-1$. For the first statement, consider an n with $\ell+1 \leq n \leq 3030-2\ell-2$. Then using $(*)$ and the inductive hypothesis, we obtain

$$v_n \geq \min(v_{n-1}+2, v_{n+1}+1) \geq \min(2\ell+2, 2\ell+2) = 2\ell+2$$

because $v_{n-1} \geq 2\ell$ (as $\ell \leq n-1 \leq 3030-2\ell$) and $v_{n+1} \geq 2\ell+1$ (as $\ell+1 \leq n+1 \leq 3030-2\ell-1$). Similarly, for $\ell+2 \leq n \leq 3030-2\ell-3$ we find

$$v_n \geq \min(v_{n-1}+2, v_{n+1}+1) \geq \min(2\ell+3, 2\ell+3) = 2\ell+3$$

because $v_{n-1} \geq 2\ell+1$ (as $\ell+1 \leq n-1 \leq 3030-2\ell-1$) and $v_{n+1} \geq 2\ell+2$ (as $\ell+1 \leq n \leq 3030-2\ell-2$). This completes the induction. □

Proof 2: The inequality $v_{1010} \geq \min(v_{1009}+2, v_{1011}+1)$ gives us two cases to consider: either $v_{1010} \geq v_{1009}+2$ or $v_{1010} \geq v_{1011}+1$.
Suppose first that $v_{1010} \geq v_{1009}+2$. In this case we will show by induction on k that $v_{1010-k} \geq v_{1009-k}+2$ for $0 \leq k \leq 1009$. The case $k=0$ is assumed, so suppose that $v_{1010-\ell} \geq v_{1009-\ell}+2$ holds for some $0 \leq \ell < 1009$. We obtain

$$v_{1009-\ell} \geq \min(v_{1010-\ell}+1, v_{1008-\ell}+2) \geq \min(v_{1009-\ell}+3, v_{1008-\ell}+2).$$

Because $v_{1009-\ell} < v_{1009-\ell}+3$ we must have $v_{1009-\ell} \geq v_{1008-\ell}+2$, completing the induction. We conclude that

$$v_{1010} \geq v_{1009}+2 \geq v_{1008}+4 \geq \cdots \geq v_0+2020 \geq 2020.$$

In the second case, we show by induction on k that $v_{1010+k} \geq v_{1011+k}+1$ for $0 \leq k \leq 2019$. Again, the base case $k=0$ is assumed, so suppose that $v_{1010+\ell} \geq v_{1011+\ell}+1$ holds for some $0 \leq \ell < 2019$. We obtain

$$v_{1011+\ell} \geq \min(v_{1010+\ell}+2, v_{1012+\ell}+1) \geq \min(v_{1011+\ell}+3, v_{1012+\ell}+1).$$

Because $v_{1011+\ell} < v_{1011+\ell}+3$, we must have $v_{1011+\ell} \geq v_{1012+\ell}+1$, completing the induction. We conclude that

$$v_{1010} \geq v_{1011}+1 \geq v_{1012}+2 \geq \cdots \geq v_{3030}+2020 \geq 2020,$$

as desired. □

Alternative E.

This solution is different from the other solutions. Shift the sequence so that it starts at a_{-1010} and ends at a_{2020}. We will show that a_0 is divisible by 2^{2020}. Consider a_0. It either has at least as many factors 2 as $2a_1$, or at least as many factors 2 as $4a_{-1}$ (this follows from $2a_1 = a_0 + 4a_{-1}$). Consider the first case, so $e_2(a_0) \geq e_2(2a_1)$. By multiplying the original recursion by 2^{n-1}, we note that $b_n = 2^n a_n$ satisfies the recursion $b_n = b_{n-1} + 8b_{n-2}$. Furthermore, we assumed that $e_2(b_0) \geq e_2(b_1)$. This implies that $e_2(b_n)$ is constant for $n \geq 1$. Furthermore, clearly $2^{2020} \mid b_{2020}$, so $2^{2020} \mid b_1$ and hence $2^{2020} \mid a_0$. The case where $e_2(a_0) \geq e_2(4a_{-1})$ is similar; we then look at $b_n = 4^n a_{-n}$ which satisfies $b_n = 8b_{n-2} - b_{n-1}$ and use that $2^{2020} \mid b_{1010}$. The rest of the argument is the same. □

Solutions to Problem 2

Answer. There are two solutions: $(\underbrace{0,0,\ldots,0}_{1010},\underbrace{1,1,\ldots,1}_{1010})$ and $(\underbrace{1,1,\ldots,1}_{1010},\underbrace{2,2,\ldots,2}_{1010})$.

Solution A. We first prove the inequality

$$((x+1)(y+1))^2 \geq 4(x^3+y^3) \tag{1}$$

for real numbers $x, y \geq 0$ satisfying $|x-y| \leq 1$, with equality if and only if $\{x,y\} = \{0,1\}$ or $\{x,y\} = \{1,2\}$.
Indeed,

$$\begin{aligned}
4(x^3+y^3) &= 4(x+y)(x^2-xy+y^2)\\
&\leq ((x+y)+(x^2-xy+y^2))^2\\
&= (xy+x+y+(x-y)^2)^2\\
&\leq (xy+x+y+1)^2\\
&= ((x+1)(y+1))^2,
\end{aligned}$$

where the first inequality follows by applying the AM-GM inequality on $x+y$ and x^2-xy+y^2 (which are clearly nonnegative). Equality holds in the first inequality precisely if $x+y = x^2-xy+y^2$ and in the second one if and only if $|x-y|=1$. Combining these equalities we have $x+y=(x-y)^2+xy = 1+xy$ or $(x-1)(y-1)=0$, which yields the solutions $\{x,y\}=\{0,1\}$ or $\{x,y\}=\{1,2\}$. Now, let $(x_1,x_2,\ldots,x_{2020})$ be any sequence satisfying conditions (i) and (ii) and let $(y_1,y_2,\ldots,y_{2020})$ be any permutation of $(x_1,x_2,\ldots,x_{2020})$. As $0 \leq \min(x_i,y_i) \leq \max(x_i,y_i) \leq \min(x_i,y_i)+1$, we can applying inequality (1) to the pair (x_i,y_i) and sum over all $1 \leq i \leq 2020$ to conclude that

$$\sum_{i=1}^{2020}((x_i+1)(y_i+1))^2 \geq 4\sum_{i=1}^{2020}(x_i^3+y_i^3) = 8\sum_{i=1}^{2020}x_i^3.$$

Therefore, in order to satisfy condition (iii), every inequality must be an equality. Hence, for every $1 \leq i \leq 2020$ we must have $\{x_i,y_i\}=\{0,1\}$ or $\{x_i,y_i\}=\{1,2\}$. By condition (ii) ,we see that either $\{x_i,y_i\}=\{0,1\}$ for all i or $\{x_i,y_i\}=\{1,2\}$ for all i.
If $\{x_i,y_i\}=\{0,1\}$ for every $1 \leq i \leq 2020$, this implies that the sequences $(x_1,x_2,\ldots,x_{2020})$ and $(y_1,y_2,\ldots,y_{2020})$ together have 2020 zeroes and 2020 ones. As $(y_1,y_2,\ldots,y_{2020})$ is a permutation of $(x_1,x_2,\ldots,x_{2020})$ this implies that $(x_1,x_2,\ldots,x_{2020})=(0,0,\ldots,0,1,1,\ldots,1)$ with 1010 zeroes and 1010 ones. Conversely, note that this sequence satisfies conditions (i), (ii), and (iii) (in (iii), we take $(y_1,y_2,\ldots,y_{2020})=(x_{2020},x_{2019},\ldots,x_1)$), showing that this sequence indeed works. The same reasoning holds for the case that $\{x_i,y_i\}=\{1,2\}$ for all i. $\square$

Comment. There are multiple ways to show the main inequality (1):

- Write

$$\begin{aligned}
((x+1)(y+1))^2 &\geq ((x+1)(y+1))^2-((x-1)(y-1))^2\\
&= 4(x+y)(xy+1)\\
&\geq 4(x+y)(xy+(x-y)^2)\\
&= 4(x^3+y^3),
\end{aligned}$$

  where equality holds precisely if $|x-y|=1$ and $(x-1)(y-1)=0$.

- Write

$$(x+1)^2(y+1)^2-4x^3-4y^3 = (x-1)^2(y-1)^2+4(x+y)\big(1-(x-y)^2\big) \geq 0,$$

  where equality holds precisely if $(x-1)(y-1)=0$ and $(x-y)^2=1$.

- One can rewrite the difference between the two sides as a sum of nonnegative expressions. One such way is to assume that $y \geq x$ and then to rewrite the difference as
$$x^2(y-2)^2 + (x+1-y)(4y^2 - 4x^2 + 2xy + x + 3y + 1),$$
where $x^2(y-2)^2 \geq 0$, $x+1-y \geq 0$ and $4y^2 - 4x^2 + 2xy + x + 3y + 1 > 0$, so in the equality case we must have $x+1-y = 0$ and $x(y-2) = 0$.

- Again assume $y \geq x$; substitute $y = x + u$ with $0 \leq u \leq 1$ and rewrite the difference as
$$x^2(x+u-2)^2 + x^2(2-2u) + (4+6u-10u^2)x + (1+2u+u^2-4u^3),$$
of which each summand is nonnegative, with equality case $u = 1$ and $x \in \{0,1\}$.

- Fix $x \geq 0$. We aim to show that the function
$$f(y) = ((x+1)(y+1))^2 - 4(x^3+y^3),$$
viewed as polynomial in y, is nonnegative on the interval $[x, x+1]$. First note that for $y = x$ and $y = x+1$ the function equals
$$(x^2-2x)^2 + 2x^2 + 4x + 1 > 0 \qquad \text{and} \qquad (x^2-x)^2 \geq 0$$
respectively. The derivative of f with respect to y equals
$$2(x+1)^2 + 2(x+1)^2 y - 12y^2,$$
which is a quadratic with negative leading coefficient that evaluates as $2(x+1)^2 > 0$ for $y = 0$. Therefore, this quadratic has one positive and one negative root. Therefore, on $[0, \infty)$, the function f will be initially increasing and eventually decreasing, hence the minimum on the interval $[x, x+1]$ will be achieved on one of the endpoints. To have equality, we must have $x^2 - x = 0$, hence $x \in \{0,1\}$.

- Observe that $((x+1)(y+1))^2 - 4(x^3+y^3)$ is the discriminant of
$$p(z) = (x^2 - xy + y^2)z^2 - (x+1)(y+1)z + (x+y).$$
Note that the leading coefficient $x^2 - xy + y^2 = (x-y)^2 + xy$ is positive unless $x = y = 0$, in which case $((0+1)^2(0+1)^2) > 4 \cdot 0^3 + 4 \cdot 0^3$. Substituting $z = 1$, we get
$$p(1) = (x^2 - xy + y^2) - (x+1)(y+1) + (x+y) = (x-y)^2 - 1 \leq 0.$$
It follows that the discriminant is non-negative. It equals zero if and only if $|x-y| = 1$ and $p(z)$ attains its minimum at $z = 1$. Without loss of generality $y = x+1$. The minimum is attained at
$$1 = \frac{(x+1)(y+1)}{2(x^2-xy+y^2)} = \frac{x^2+3x+2}{2x^2+2x+2},$$
which reduces to $x^2 = x$. Therefore, the only critical points are $(x,y) = (0,1)$ and $(x,y) = (1,2)$.

- Without loss of generality $y \geq x$. Let $z = \frac{x+y}{2}$ and $a = y - z$. Note that $z \geq 0$ and $0 \leq a \leq \min\{\frac{1}{2}, z\}$. We can rewrite the inequality in terms of z and a.
$$\begin{aligned}((x+1)(y+1))^2 - 4x^3 - 4y^3 &= ((z+1+a)(z+1-a))^2 - 4(z-a)^3 - 4(z+a)^3 \\ &= ((z+1)^2 - a^2)^2 - 8z^3 - 24za^2 \\ &= a^4 - (24z + 2(z+1)^2)a^2 + ((z+1)^4 - 8z^3)\end{aligned}$$
This is a quadratic in a^2. It attains its minimum at $a^2 = 12z + (z+1)^2 \geq 1$. Therefore it is strictly decreasing in a on the interval $[0, \min\{\frac{1}{2}, z\}]$. If $a = \frac{1}{2}$, then $y = x+1$. It follows that
$$((x+1)(y+1))^2 - 4x^3 - 4y^3 = ((x+1)(x+2))^2 - 4x^3 - 4(x+1)^3 = x^2(x-1)^2 \geq 0$$
with equality if and only if $x = 0$ or $x = 1$. If $a = z$, then $x = 0$. It follows that
$$((x+1)(y+1))^2 - 4x^3 - 4y^3 = (y+1)^2 - 4y^3 = y^2(1-y) + 2y(1-y^2) + (1-y^3) \geq 0$$
with equality if and only if $y = 1$. Therefore $(0,1)$ and $(1,2)$ are the only critical points.

Solutions to Problem 3

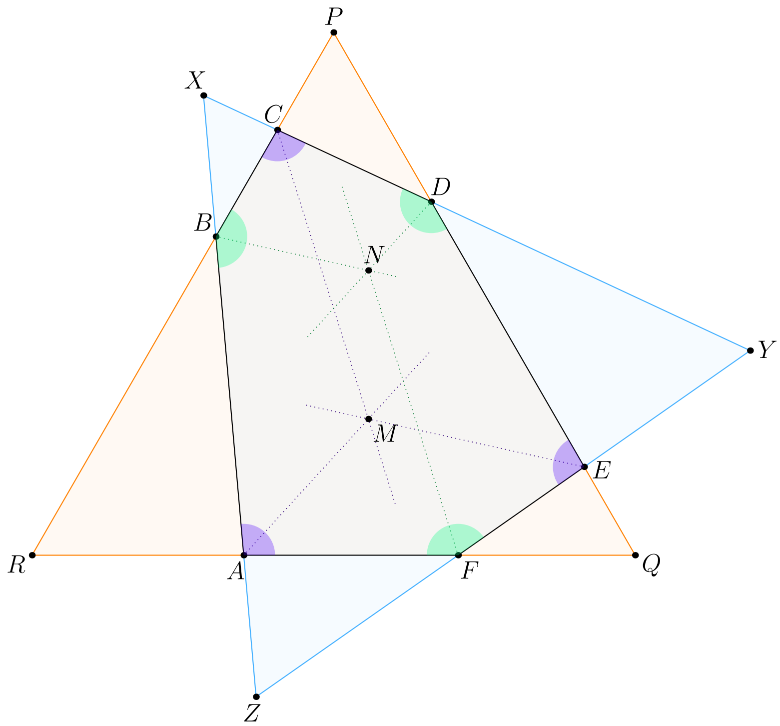
Solution A. Denote the angle bisector of A by a and similarly for the other bisectors. Thus, given that a, c, e have a common point M, we need to prove that b, d, f are concurrent. We write $\angle(x, y)$ for the value of the directed angle between the lines x and y, i.e. the angle of the counterclockwise rotation from x to y (defined $\pmod{180°}$).
Since the sum of the angles of a convex hexagon is $720°$, from the angle conditions we get that the sum of any two consecutive angles is equal to $240°$. In particular, it now follows that $\angle(b, a) = \angle(c, b) = \angle(d, c) = \angle(e, d) = \angle(f, e) = \angle(a, f) = 60°$ (assuming the hexagon is clockwise oriented).
Let $X = AB \cap CD$, $Y = CD \cap EF$ and $Z = EF \cap AB$. Similarly, let $P = BC \cap DE$, $Q = DE \cap FA$ and $R = FA \cap BC$. From $\angle B + \angle C = 240°$ it follows that $\angle(ZX, XY) = \angle(BX, XC) = 60°$. Similarly we have $\angle(XY, YZ) = \angle(YZ, ZX) = 60°$, so triangle XYZ (and similarly triangle PQR) is equilateral. We see that the hexagon $ABCDEF$ is obtained by intersecting the two equilateral triangles XYZ and PQR.
We have $\angle(AM, MC) = \angle(a, c) = \angle(a, b) + \angle(b, c) = 60°$, and since

$$\angle(AM, MC) = \angle(AX, XC) = \angle(AR, RC) = 60°,$$

A, C, M, X, R are concyclic. Because M lies on the bisector of angle $\angle XAR$, we must have $MR = MX$, so triangle MRX is isosceles. Moreover, we have $\angle(MR, MX) = \angle(CR, CX) = \angle(BC, CD)$, which is angle C of the hexagon. We now see that the triangles MRX, MPY and MQZ are isosceles and similar. This implies that there is a rotation centered at M that sends X, Y and Z to R, P and Q respectively. In particular, the equilateral triangles XYZ and PQR are congruent.
It follows that there also exists a rotation sending X, Y, Z to P, Q, R respectively. Define N as the center of this rotation. Triangles NXZ and NPR are congruent and equally oriented, hence N is equidistant from XZ and PR and lies on the inner bisector b of $\angle B$ (we know N lies on the inner, not the outer bisector because the rotation centered at N is clockwise). In the same way we can show that N is on d and on f, so b, d, f are concurrent at N. □

Remark. The key observation (a rotation centered at M sends $\triangle XYZ$ to $\triangle RQP$) can be established in slightly different ways. E.g., since A, M, R, X are concyclic and A, M, Q, Z are concyclic, M is the Miquel point of the lines XZ, RQ, XR, ZQ, hence it is the center of similitude s sending $\vec{XZ}$ to $\vec{RQ}$. Repeating the same argument for the other pairs of vectors, we obtain that s sends $\triangle XYZ$ to $\triangle RQP$. Moreover, s is a rotation, since M is equidistant from XZ and RQ.

Remark. The reverse argument can be derived in a different way, e.g., defining N as the common point of the circles $BXPD$, $DYQF$, $FZRB$, and showing that $\triangle NXZ = \triangle NPR$, etc.



Solution B. As in Solution A, we prove that the hexagon $ABCDEF$ is the intersection of the equilateral triangles PQR and XYZ.
Let $d(S, AB)$ denote the signed distance from the point S to the line AB, where the negative sign is taken if AB separates S and the hexagon. We define similarly the other distances ($d(S, BC)$, etc). Since $M \in a$, we have $d(M, ZX) = d(M, QR)$. In the same way, we have $d(M, XY) = d(M, RP)$ and $d(M, YZ) = d(M, PQ)$. Therefore $d(M, ZX) + d(M, XY) + d(M, YZ) = d(M, QR) + d(M, RP) + d(M, PQ)$.
We now use of the following well-known lemma (which can be easily proved using areas) to deduce that triangles PQR and XYZ are congruent.

Lemma. The sum of the signed distances from any point to the sidelines of an equilateral triangle (where the signs are taken such that all distances are positive inside the triangle) is constant and equals the length of the altitude.

For $N = b \cap d$ we now find $d(N, ZX) = d(N, RP)$ and $d(N, XY) = d(N, PQ)$. Using again the lemma for the point N, we get $d(N, ZX) + d(N, XY) + d(N, YZ) = d(N, QR) + d(N, RP) + d(N, PQ)$. Therefore $d(N, YZ) = d(N, QR)$, thus $N \in f$. □

Remark. Instead of using the lemma, it is possible to use some equivalent observation in terms of signed areas.

Solution C. We use the same notations as in Solution A. We will show that a, c and e are concurrent if and only if

$$AB + CD + EF = BC + DE + FA,$$

which clearly implies the problem statement by symmetry.
Let $\vec{a}$ be the vector of unit length parallel to a directed from A towards the interior of the hexagon. We define analogously $\vec{b}$, etc. The angle conditions imply that opposite bisectors of the hexagon are parallel, so we have $\vec{a} \parallel \vec{d}$, $\vec{b} \parallel \vec{e}$ and $\vec{c} \parallel \vec{f}$. Moreover, as in the previous solutions, we know that $\vec{a}$, $\vec{c}$ and $\vec{e}$ make angles of $120°$ with each other. Let $M_A = c \cap e$, $M_C = e \cap a$ and $M_E = a \cap c$. Then M_A, M_C, M_E form an equilateral triangle with side length denoted by s. Note that the case $s = 0$ is equivalent to a, c and e being concurrent.
Projecting $\vec{M_EA} + \vec{AB} = \vec{M_EB} = \vec{M_EC} + \vec{CB}$ onto $\vec{e} = -\vec{b}$, we obtain

$$\vec{AB} \cdot \vec{b} - \vec{CB} \cdot \vec{b} = \vec{M_EC} \cdot \vec{b} - \vec{M_EA} \cdot \vec{b} = \vec{M_EA} \cdot \vec{e} - \vec{M_EC} \cdot \vec{e}.$$

Writing $\varphi = \frac{1}{2}\angle B = \frac{1}{2}\angle D = \frac{1}{2}\angle F$, we know that $\vec{AB} \cdot \vec{b} = -AB \cdot \cos(\varphi)$, and similarly $\vec{CB} \cdot \vec{b} = -CB \cdot \cos(\varphi)$. Because M_EA and M_EC intersect e at $120°$ angles, we have $\vec{M_EA} \cdot \vec{e} = \frac{1}{2}M_EA$ and $\vec{M_EC} \cdot \vec{e} = \frac{1}{2}M_EC$. We conclude that

$$2\cos(\varphi)(AB - CB) = M_EC - M_EA.$$

Adding the analogous equalities $2\cos(\varphi)(CD - ED) = M_AE - M_AC$ and $2\cos(\varphi)(EF - AF) = M_CA - M_CE$, we obtain

$$2\cos(\varphi)(AB + CD + EF - CB - ED - AF) = M_EC - M_EA + M_AE - M_AC + M_CA - M_CE.$$

Because M_A, M_C and M_E form an equilateral triangle with side length s, we have $M_EC - M_AC = \pm s$, $M_CA - M_EA = \pm s$, and $M_AE - M_CE = \pm s$. Therefore, the right hand side $M_EC - M_EA + M_AE - M_AC + M_CA - M_CE$ equals $\pm s \pm s \pm s$, which (irrespective of the choices of the $\pm$-signs) is 0 if and only if $s = 0$. Because $\cos(\varphi) \neq 0$, we conclude that

$$AB + CD + EF = CB + ED + AF \iff s = 0 \iff a,\ c,\ e \text{ concurrent},$$

as desired. □

Remark. Equalities used in the solution could appear in different forms, in particular, in terms of signed lengths.

Remark. Similar solutions could be obtained by projecting onto the line perpendicular to b instead of b.

Solution D. We use the the same notations as in previous solutions and the fact that $a \parallel d$, $b \parallel e$ and $c \parallel f$ make angles of $120°$. Also, we may assume that E and C are not symmetric in a (if they are, the entire figure is symmetric and the conclusion is immediate).
We consider two mappings: the first one $s : a \to BC \to d$ sending $A' \mapsto B' \mapsto S$ is defined such that $A'B' \parallel AB$ and $B'S \parallel b$, and the second one $t : a \to EF \to d$ sending $A' \mapsto F' \mapsto T$ is defined such that $A'F' \parallel AF$ and $F'T \parallel f$. Both maps are affine linear since they are compositions of affine tranformations. We will prove that they coincide by finding two distinct points $A', A'' \in a$ for which $s(A') = t(A')$ and $s(A'') = t(A'')$. Then we will obtain that $s(A) = t(A)$, which by construction implies that the bisectors of $\angle B$, $\angle D$ and $\angle F$ are concurrent.
We will choose A' to be the reflection of C in e and A'' to be the reflection of E in c. They are distinct since otherwise C and E would be symmetric in a. Applying the above maps $a \to BC$ and $a \to EF$ to A', we get points B' and F' such that $A'B'CDEF'$ satisfies the problem statement. However, this hexagon is symmetric in e, hence the bisectors of $\angle B'$, $\angle D$, $\angle F'$ are concurrent and $s(A') = t(A')$. The same reasoning yields $s(A'') = t(A'')$, which finishes the solution. □

Remark. This solution is based on the fact that that two specific affine linear maps coincide. Here it was proved by exhibiting two points where they coincide. One could prove it in another way, exhibiting one such point and proving that the 'slopes' are equal.

Remark. There are similar solutions where claims and proofs could be presented in more 'elementary' terms. For example, an elementary reformulation of the 'slopes' being equal is: if b' passes through B' parallel to b, and f' passes through F' parallel to f, then the line through $b \cap f$ and $b' \cap f'$ is parallel to a (which is parallel to d).

Solution E. We use the same notations as in previous solutions.
Since the sum of the angles of a convex hexagon is $720°$, from the angle conditions we get $\angle B + \angle C = 720°/3 = 240°$. From $\angle B + \angle C = 240°$ it follows that the angle between c and b equals $60°$. The same is analogously true for other pairs of bisectors of neighboring angles.
Consider the points $O_a \in a$, $O_c \in c$, $O_e \in e$, each at the same distance d' from M, where $d' > \max\{MA, MC, ME\}$, and such that the rays AO_a, CO_c, EO_e point out of the hexagon. By

construcion, O_a and O_c are symmetrical in e, hence $O_aO_c \perp b$. Similarly, $O_cO_e \perp d$, $O_eO_a \perp f$. Thus it suffices to prove that perpendiculars from B, D, F to the sidelines of $\triangle O_aO_cO_e$ are concurrent. By a well-known criteria, this condition is equivalent to equality

$$O_aB^2 - O_cB^2 + O_cD^2 - O_eD^2 + O_eF^2 - O_aF^2 = 0. \qquad (*)$$

To prove $(*)$ consider a circle ω_a centered at O_a and tangent to AB and AF and define circles ω_c and ω_e in the same way. Rewrite O_aB^2 as $r_a^2 + B_aB^2$, where r_a is the radius of ω_a, and B_a is the touch point of ω_a with AB. Using similar notation for the other tangent points, transform $(*)$ into

$$B_aB^2 - B_cB^2 + D_cD^2 - D_eD^2 + F_eF^2 - F_aF^2 = 0. \qquad (**)$$

Furthermore, $\angle O_cO_aB_a = \angle MO_aB_a + \angle O_cO_aM = (90^\circ - \varphi) + 30^\circ = 120^\circ - \varphi$, where $\varphi = \frac{1}{2}\angle A$. (Note that $\varphi > 30^\circ$, since $ABCDEF$ is convex.) By analogous arguments, $\angle O_aO_cB_c = \angle O_eO_cD_c = \angle O_cO_eD_e = \angle O_aO_eF_e = \angle O_eO_aF_a = 120^\circ - \varphi$. It follows that rays O_aB_a and O_cB_c (being symmetrical in e) intersect at $U_e \in e$ forming an isosceles triangle $\triangle O_aU_eO_c$. Similarly define $\triangle O_cU_aO_e$ and $\triangle O_eU_cO_a$. These triangles are congruent (equal bases and corresponding angles). Therefore we have $O_aU_c = U_cO_e = O_eU_a = U_aO_c = O_cU_e = U_eO_a$. Moreover, we also have $B_aU_e = O_aU_e - r_a = O_aU_c - r_a = F_aU_c = x$, and thus similarly $D_cU_a = B_cU_e = y$, $F_eU_c = D_eU_a = z$.

Now from quadrilateral $BB_aU_eB_c$ with two opposite right angles $B_aB^2 - B_cB^2 = B_cU_e^2 - B_aU_e^2 = y^2 - x^2$. Similarly $D_cD^2 - D_eD^2 = D_eU_a^2 - D_cU_a^2 = z^2 - y^2$ and $F_eF^2 - F_aF^2 = F_aU_c^2 - F_eU_c^2 = x^2 - z^2$. Finally, we substitute this into $(**)$, and the claim is proved. □

Remark. Circles ω_a, ω_c and ω_e could be helpful in some other solutions. In particular, the movement of A along a in Solution D is equivalent to varying r_a.

Solutions to Problem 4

Solution A. Let $\sigma = (\sigma_1, \ldots, \sigma_{n-1})$ be a fresh permutation of the integers $1, 2, \ldots, n-1$. We claim that for any $1 \le i \le n-1$ the permutation

$$\sigma^{(i)} = (\sigma_1, \ldots, \sigma_{i-1}, n, \sigma_i, \ldots, \sigma_{n-1})$$

is a fresh permutation of the integers $1, 2, \ldots, n$. Indeed, let $1 \le k \le n-1$. If $k \ge i$ then we have $n \in \{\sigma_1^{(i)}, \ldots, \sigma_k^{(i)}\}$, but $n \notin \{1, 2, \ldots, k\}$. And if $k < i$ we have $k < n-1$, and $\{\sigma_1^{(i)}, \ldots, \sigma_k^{(i)}\} = \{\sigma_1, \ldots, \sigma_k\} \neq \{1, 2, \ldots, k\}$, since σ is fresh. Moreover, it is easy to see that when we apply this construction to all fresh permutations of $1, 2, \ldots, n-1$, we obtain $(n-1) \cdot f_{n-1}$ distinct fresh permutations of $1, 2, \ldots, n$.
Note that a fresh permutation of $1, 2, \ldots, n-1$ cannot end in $n-1$, and hence none of the previously constructed fresh permutations of $1, 2, \ldots, n$ will end in $n-1$ either. Therefore, we will complete the proof by finding f_{n-1} fresh permutations of $1, 2, \ldots, n$ that end in $n-1$. To do this, let $\sigma = (\sigma_1, \ldots, \sigma_{n-1})$ be a fresh permutation of $1, 2, \ldots, n-1$ and let j be such that $\sigma_j = n-1$. Define

$$\sigma' = (\sigma_1, \ldots, \sigma_{j-1}, n, \sigma_{j+1}, \ldots, \sigma_{n-1}, n-1),$$

then clearly σ' is a permutation of $1, 2, \ldots, n$ that ends in $n-1$. We show that σ' is fresh, so let $1 \le k \le n-1$. If $k \ge j$ then $n \in \{\sigma'_1, \ldots, \sigma'_k\}$ but $n \notin \{1, 2, \ldots, k\}$; if $k < j$, then $k < n-1$ and $\{\sigma'_1, \ldots, \sigma'_k\} = \{\sigma_1, \ldots, \sigma_k\} \neq \{1, 2, \ldots, k\}$, since σ is fresh. So we have constructed f_{n-1} additional fresh permutations of 1, 2, ..., n (which again are all different), and the total number f_n of fresh permutations of 1, 2, ..., n must at least be $(n-1)f_{n-1} + f_{n-1} = nf_{n-1}$, as required. □

Comment. A similar way to construct n fresh permutations of 1, 2, ..., n for each fresh permutation σ of 1, 2, ..., $n-1$ is as follows: increase all entries of σ by 1, and then add the number 1 anywhere; all these permutations are fresh, except the ones where 1 is added at the front, which can be made fresh by swapping the 1 and the 2. It is again straightforward to check that we obtain nf_{n-1} permutations that are fresh and distinct, although a little extra care is needed to account for the fact that we increased the entries of our original permutation.

Solution B. Assuming $n \ge 3$, we construct $f_{n-1} \cdot n$ different fresh permutations.
Consider a fresh permutation of the $n-1$ numbers $1, 3, 4, \ldots, n$ (the number 2 has been removed), by which we mean a permutation $(x_1, \ldots, x_{n-1})$ such that $x_1 \neq 1$ and $\{x_1, \ldots, x_k\} \neq \{1, 3, \ldots, k+1\}$ for all k with $2 \le k \le n-2$. There are exactly f_{n-1} such permutations.
By inserting 2 anywhere in such a permutation, i.e. before or after all entries, or between two entries x_i, x_{i+1}, we generate the following list of n distinct permutations of $1, 2, \ldots, n$, which we claim are all fresh:

$$(2, x_1, \ldots, x_{n-1}), \quad \ldots, \quad (x_1, \ldots, x_{i-1}, 2, x_i, \ldots, x_{n-1}), \quad \ldots, \quad (x_1, \ldots, x_{n-1}, 2).$$

In order to verify freshness, suppose that some permutation above is not fresh, that is, for some $1 \le k \le n-1$, the first k elements are $1, \ldots, k$. If $k = 1$ then the first element is 1; but the first element is either 2 or x_1 and $x_1 \neq 2$, so this is not possible. If $k \ge 2$ then the first k elements must contain 2 and $x_1, \ldots, x_{k-1}$, so $\{x_1, \ldots, x_{k-1}\} = \{1, 3, \ldots, k\}$, in contradiction to the fact that $(x_1, \ldots, x_{n-1})$ is fresh.
We have thus constructed $f_{n-1} \cdot n$ different fresh permutations, so $f_n \ge f_{n-1} \cdot n$. □

Comment. Note that the construction from solution B can also be performed by placing $n-1$ instead of 2, or indeed any k between 2 and $n-1$ (in which case it will work for $n \ge k+1$).
Moreover, similar arguments can be made while choosing to work with non-fresh permutations rather than fresh permutations. Indeed, in order to show that $n! - f_n \le n((n-1)! - f_{n-1})$ for all $n \ge 3$, which is equivalent to the problem statement, one can argue as follows. Given a non-fresh permutation $(\sigma_1, \ldots, \sigma_n)$ of $1, 2, \ldots, n$, erasing $n-1$ from it and changing n to $n-1$ yields a permutation which cannot be fresh. Moreover, each non-fresh permutation of $1, 2, \ldots, n-1$ can be obtained in this way from at most n different non-fresh permutations of $1, 2, \ldots, n$. This type of argument would essentially mirror the one carried out in Solution B.

Solution C. By considering all permutations of $1, 2, \ldots, n$ and considering the smallest k for which the first k numbers are $1, 2, \ldots, k$ in some order, one can deduce the recursion

$$\sum_{k=1}^{n} (n-k)! \cdot f_k = n!,$$

which holds for any $n \geq 1$.
Therefore, if $n \geq 3$ we have

$$\begin{aligned}
0 &= n! - (n+1)(n-1)! + (n-1)(n-2)! \\
&= \sum_{k=1}^{n} (n-k)! \cdot f_k - (n+1) \sum_{k=1}^{n-1} (n-1-k)! \cdot f_k + (n-1) \sum_{k=1}^{n-2} (n-2-k)! \cdot f_k \\
&= f_n - n f_{n-1} + \sum_{k=1}^{n-2} (n-2-k)! \cdot ((n-k)(n-k-1) - (n+1)(n-k-1) + (n-1)) \cdot f_k \\
&= f_n - n f_{n-1} + \sum_{k=1}^{n-2} (n-2-k)! \cdot k(k+2-n) \cdot f_k \leq f_n - n f_{n-1},
\end{aligned}$$

which rewrites as $f_n \geq n f_{n-1}$. □

Solution D. We first show that for any $n \geq 2$ we have

$$f_n = \sum_{k=1}^{n-1} k(n-1-k)! \cdot f_k.$$

To deduce this relation, we imagine obtaining permutations of $1, 2, \ldots, n$ by inserting n into a permutation of the numbers $1, 2, \ldots, n-1$. Specifically, let $(\sigma_1, \ldots, \sigma_{n-1})$ be *any* permutation of $1, 2, \ldots, n-1$, and let $1 \leq k \leq n-1$ be minimal with $\{\sigma_1, \ldots, \sigma_k\} = \{1, 2, \ldots, k\}$. Note that there are $(n-1-k)! \cdot f_k$ such permutations. Furthermore, inserting n in this permutation will give a fresh permutation if and only if n is inserted before σ_k, so there are k options to do so. Consequently, if $n \geq 3$,

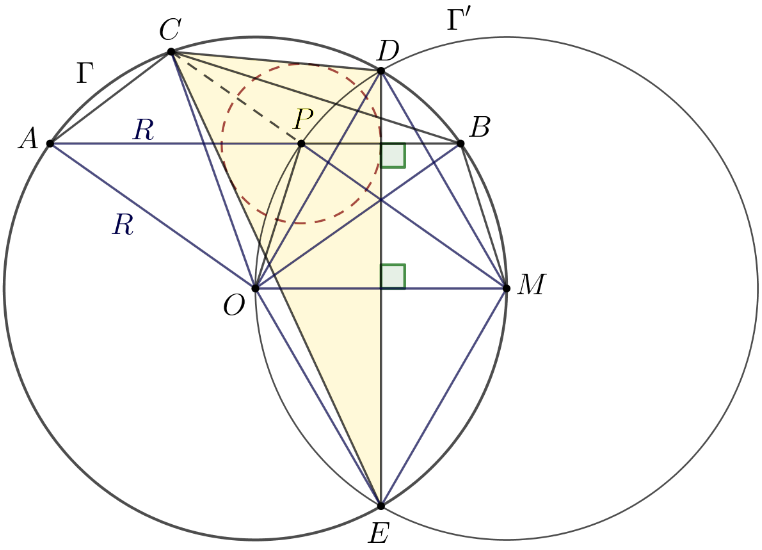
$$\begin{aligned}
f_n &= \sum_{k=1}^{n-1} k(n-1-k)! \cdot f_k \\
&= (n-1) \cdot f_{n-1} + \sum_{k=1}^{n-2} k(n-1-k)! \cdot f_k \\
&\geq (n-1) \cdot f_{n-1} + \sum_{k=1}^{n-2} k(n-2-k)! \cdot f_k \\
&= (n-1) \cdot f_{n-1} + f_{n-1} = n \cdot f_{n-1},
\end{aligned}$$

completing the proof. □

Comment. Fresh permutations are known as *indecomposable permutations* or *irreducible permutations* in the literature. The problem asks to prove that the probability that a randomly chosen permutation of 1, 2, …, n is indecomposable is a non-decreasing function of n. In fact, it turns out that this probability goes to 1 as $n \to \infty$: for large n, almost all permutations of 1, 2, …, n are indecomposable. More can be found in:
Y. Koh & S. Ree. Connected permutation graphs. *Discrete Mathematics* **307** (21):2628–2635, 2007.

Solutions to Problem 5

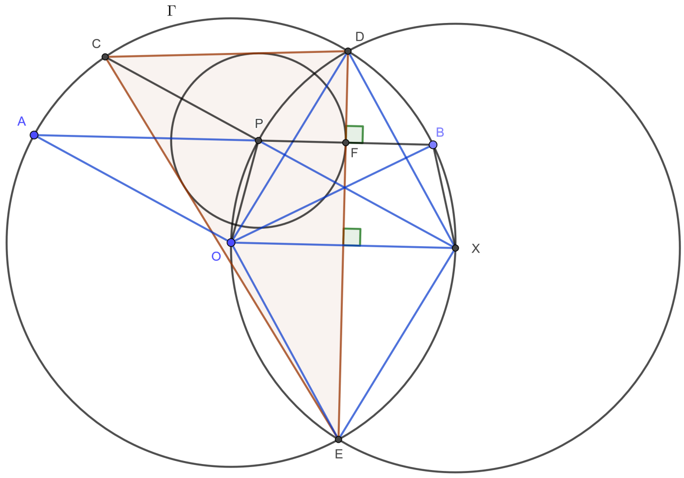
Solution A. The angle bisector of $\angle ECD$ intersects the circumcircle of CDE (which is Γ) at the midpoint M of arc DBE. It is well-known that the incentre is the intersection of the angle bisector segment CM and the circle with centre at M and passing through D, E. We will verify this property for P.



By the conditions we have $AP = OA = OB = OC = OD = OE = R$. Both lines OM and APB are perpendicular to ED, therefore $AP \| OM$; in the quadrilateral $AOMP$ we have $AP = OA = AM = R$ and $AP \| OM$, so $AOMP$ is a rhombus and its fourth side is $PM = R$.
In the convex quadrilateral $OMBP$ we have $OM \| PB$, so $OMBP$ is a symmetric trapezoid; the perpendicular bisector of its bases AO and PB coincide. From this symmetry we obtain $MD = OD = R$ and $ME = OE = R$. (Note that the triangles OEM ad OMD are equilateral.)
We already have $MP = MD = ME = R$, so P indeed lies on the circle with center M and passing through D, E. (Notice that this circle is the reflection of Γ about DE.)

From $PB = PC$ and $OB = OC$ we know that B and C are symmetrical about OP; from the rhombus $AOMP$ we find that A and M are also symmetrical about OP. By reflecting the collinear points B, P, A (with P lying in the middle) we get that C, P, M are collinear (and P is in the middle). Hence, P lies on the line segments CM. $\square$

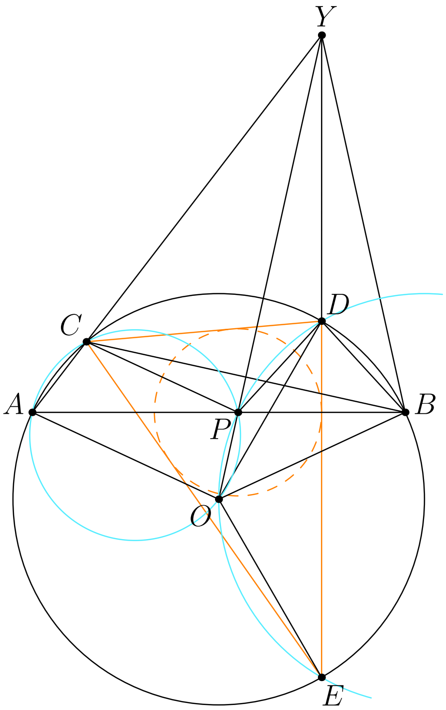
Solution B. Let X be the second intersection of CP with Γ. Using the power of the point P in the circle Γ and the fact that $PB = PC$, we find that $PX = PA = R$. The quadrilateral $AOXP$ has four sides of equal length, so it is a rhombus and in particular OX is parallel to AP. This proves that $OXBP$ is a trapezoid, and because the diagonals PX and OB have equal length, this is even an isosceles trapezoid. Because of that, DE is not only the perpendicular bisector of PB, but also of OX.



In particular we have $XD = XP = XO = XE = R$, which proves that X is the middle of the arc DE and P belongs to the circle with center X going through D and E. These properties, together with the fact that C, P, X are collinear, determine uniquely the incenter of CDE.

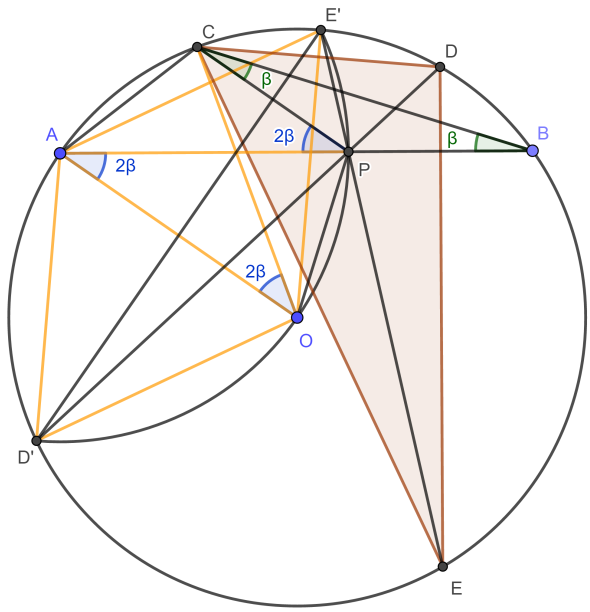
□

Solution C. Let Y be the circumcenter of triangle BPC. Then from $YB = YP$ it follows that Y lies on DE (we assume D lies in between Y and E), and from $YB = YC$ it follows that Y lies on OP, where O is the center of Γ.
From $\angle AOC = 2\angle ABC = \angle APC$ (because $\angle PBC = \angle PCB$) we deduce that $AOPC$ is a cyclic quadrilateral, and from $AP = R$ it follows that $AOPC$ is an isosceles trapezoid. We now find that $\angle YCP = \angle YPC = 180° - \angle OPC = 180° - \angle ACP$, so Y lies on AC.



Power of a point gives $YO \cdot YP = YC \cdot YA = YD \cdot YE$, so D, P, O and E are concyclic. It follows that $2\angle DAE = \angle DOE = \angle DPE = \angle DBE = 180° - \angle DAE$, so $\angle DAE = 60°$. We can now finish the proof by angle chasing.
From $AB \perp DE$ we have $\angle AOD + \angle BOE = 180°$ and from $\angle DOE = 2\angle DAE = 120°$ it follows that $\angle BOD + \angle BOE = 120°$. It follows that $\angle AOD - \angle BOD = 180° - 120° = 60°$. Let $\angle OAB = \angle OBA = 2\beta$; then $\angle AOD + \angle BOD = \angle AOB = 180° - 4\beta$. Together with $\angle AOD - \angle BOD = 60°$, this yields $\angle AOD = 120° - 2\beta$ and $\angle BOD = 60° - 2\beta$. We now find $\angle AED = \frac{1}{2}\angle AOD = 60° - \beta$, which together with $\angle DAE = 60°$ yields $\angle ADE = 60° + \beta$. From the isosceles trapezoid $AOPC$ we have $\angle CDA = \angle CBA = \frac{1}{2}\angle CPA = \frac{1}{2}\angle PAO = \beta$, so $\angle CDE = \angle CDA + \angle ADE = \beta + 60° + \beta = 60° + 2\beta$.
From $\angle BOD = 60° - 2\beta$ we deduce that $\angle BED = 30° - \beta$; together with $\angle DBE = 120°$ this yields $\angle EDB = 30° + \beta$. We now see that $\angle PDE = \angle BDE = 30° + \beta = \frac{1}{2}\angle CDE$, so P is on the angle bisector of $\angle CDE$. Similarly, P lies on the angle bisector of $\angle CED$, so P is the incenter of CDE. $\square$

Solution D. We draw the lines DP and EP and let D' resp. E' be the second intersection point with Γ.



The triangles APD' and DPB are similar, and the triangles APE' and EPB are also similar, hence they are all isosceles and it follows that E', O, P, D' lie on a circle with center A. In particular AOD' and AOE' are equilateral triangles. Angle chasing gives

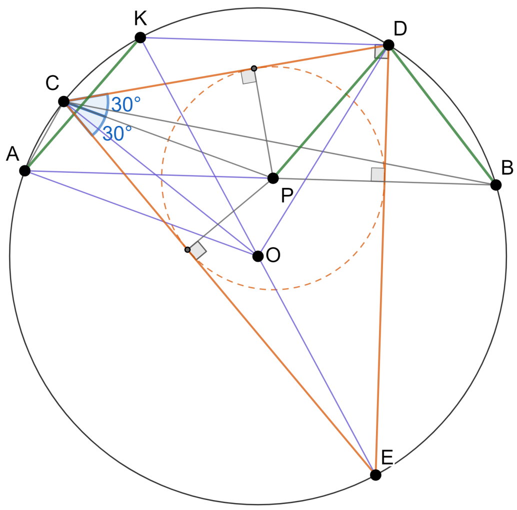
$$\angle CDP = \angle CDD' = \frac{1}{2}\angle COD' = \frac{1}{2}(60^\circ + \angle COA)$$

$$\angle EDP = \angle EDD' = \angle EE'D' = \angle PE'D' = \frac{1}{2}\angle PAD' = \frac{1}{2}(60^\circ + \angle PAO)$$

Similarly we prove $\angle CEP = \frac{1}{2}(60^\circ - \angle COA)$ and $\angle DEP = \frac{1}{2}(60^\circ - \angle PAO)$ so if we can prove that $\angle COA = \angle PAO$, we will have proven that P belongs to the angle bisector of $\angle CED$ and to the angle bisector of $\angle CDE$, which is enough to prove that P is the incenter of the triangle CDE.
Let $\beta = \angle ABC = \angle PCB$. We have $\angle APC = 2\beta$ and $\angle AOC = 2\beta$, so $AOPC$ is an inscribed quadrilateral. Moreover, since the diagonals AP and CO have equal length, this is actually an isosceles trapezoid, and hence $\angle PAO = \angle CPA = \angle COA$ which concludes the proof.

□

Solution E. Without loss of generality we assume that D and C are in the same half-plane regarding line AB.
Since $PC = BP$ and $ABCD$ is inscribed quadrilateral we have $\angle PCB = \angle CBP = \angle CEA = \alpha$. As in the other solutions, $AOPC$ is an isosceles trapezoid and $2\alpha = \angle CPA = \angle PCO$

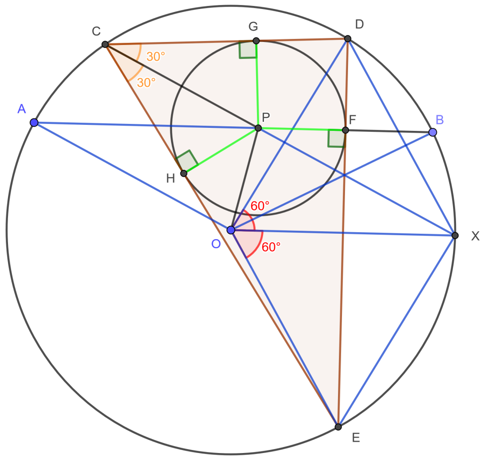


Let K be intersection of EO and Γ. Then $\angle KDE = 90^o$, $AB \| DK$ and $KDBA$ is isosceles trapezoid. We obtain $DP = BD = AK$, which implies that $DPAK$ is a parallelogram and hence $DK = PA = R = OD = OK$. We see that DOK is an equilateral triangle. Then $\angle ECD = \angle EKD = 60^o$.
Further we prove that PC bisects $\angle ECD$ using $\angle DCB = \angle DKB = \angle KDA$ (from isosceles trapeziod $DKAB$) and that $\angle KEC = \angle OEC = \angle OCE$ (from isosceles triangle OCE):

$$\angle DCP = \angle DCB + \angle BCP = \angle KDA + \alpha = \angle KEC + \angle CEA + \alpha = \angle OCE + 2\alpha = \angle PCE$$

Further by $\angle DCP = \angle PCE = \frac{1}{2}\angle ECD = 30^\circ$ distances between P and sides $\triangle CDE$ are $\frac{1}{2}PC = \frac{1}{2}PB$ (as ratio between cathetus and hypotenuse in right triangle with angles 60^o and 30^o). So, we have found incentre. $\square$

Solution F. Let F, G, H be the projections of P on the sides DE, DC resp. CE. If P is indeed the incenter, then the three line segments PF, PG, PH have the same length. This means that the problem is equivalent to proving that $PG = PH = PF = \frac{1}{2}PB = \frac{1}{2}PC$ and thus trigonometry in the right-angled triangles CPG and CPH tells us that it is enough to prove that $\angle DCP = \angle ECP = 30°$.



We introduce the point X as the second intersection of the line CP with Γ. Because O is the center of Γ we can reduce the problem to proving that $\angle XOD = \angle XOE = 60°$, or equivalently that XOD and XOE are equilateral triangles. This last condition is equivalent to X being the reflection of O on the line DE. Following the chain of equivalences, we see therefore that in order to solve the problem it is enough to prove that X is the reflection of O on DE. We prove this property as in Solution B, using the fact that $OXBP$ is an isosceles trapezoid.

□

Solution G. Assume AB is parallel to the horizontal axis, and that Γ is the unit circle. Write $f(\theta)$ for the point $(\cos(\theta), \sin(\theta))$ on Γ. As in Solution C, assume that $\angle AOB = 180° - 4\beta$; then we can take $B = f(2\beta)$ and $A = f(180° - 2\beta)$. As in Solution C, we observe that $AOPC$ is an isosceles trapezoid, which we use to deduce that $\angle ABC = \frac{1}{2}\angle APC = \frac{1}{2}\angle OAB = \beta$. We now know that $C = f(180° - 4\beta)$.

The point P lies on AB with $AP = R = 1$, so $P = (\cos(180° - 2\beta) + 1, \sin(2\beta)) = (1 - \cos(2\beta), \sin(2\beta))$. The midpoint of BP therefore has coordinates $(\frac{1}{2}, \sin(2\beta))$, so D and E have x-coordinate $\frac{1}{2}$. Without loss of generality, we take $D = f(60°)$ and $E = f(-60°)$.

We have now obtained coordinates for all points in the problem, with one free parameter (β). To show that P is the incenter of CDE, we will show that P lies on the bisector of $\angle CDE$; analogously, one can show that P lies on the bisector of angle CED. The bisector of angle CDE passes through the midpoint M of the arc CE not containing DE; because $C = f(180° - 4\beta)$ and $E = f(-60°)$, we have $M = f(240° - 2\beta)$.

It remains to show that $P = (1 - \cos(2\beta), \sin(2\beta))$ lies on the line connecting the points $D = (\cos(60°), \sin(60°))$ and $M = (\cos(240° - 2\beta), \sin(240° - 2\beta)) = (-\cos(60° - 2\beta), -\sin(60° - 2\beta))$. The equation for the line DM is

$$(Y + \sin(60° - 2\beta))(\cos(60°) + \cos(60° - 2\beta)) = (\sin(60°) + \sin(60° - 2\beta))\,(X + \cos(60° - 2\beta))\,,$$

which, using the fact that $\cos(60°) + \cos(60° - 2\beta) = 2\cos(\beta)\cos(60° - \beta)$ and $\sin(60°) + \sin(60° - 2\beta) = 2\cos(\beta)\sin(60° - \beta)$, simplifies to

$$(Y + \sin(60° - 2\beta))\cos(60° - \beta) = (X + \cos(60° - 2\beta))\sin(60° - \beta).$$

Because $\cos(60° - 2\beta)\sin(60° - \beta) - \sin(60° - 2\beta)\cos(60° - \beta) = \sin(\beta)$, this equation further simplifies to

$$Y\cos(60° - \beta) - X\sin(60° - \beta) = \sin(\beta).$$

Plugging in the coordinates of P, i.e., $X = 1 - \cos(2\beta)$ and $Y = \sin(2\beta)$, shows that P is on this line: for this choice of X and Y, the left hand side equals $\sin(60^\circ + \beta) - \sin(60^\circ - \beta) = 2\cos(60^\circ)\sin(\beta)$, which is indeed equal to $\sin(\beta)$. So P lies on the bisector DM of $\angle CDE$, as desired. $\square$

Solution H. Let Γ be the complex unit circle and let AB be parallel with the real line and $0 < \varphi = \arg b < \frac{\pi}{2}$. Then

$$|b| = 1, \quad a = -\bar{b}, \quad p = a + 1 = 1 - \bar{b}.$$

From $\operatorname{Re} d = \operatorname{Re} e = \operatorname{Re} \frac{p+b}{2} = \frac{1}{2}$ we get that $d = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $e = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ are conjugate 6th roots of unity; $d^3 = e^3 = -1$, $d + e = 1$, $d^2 = -e$, $e^2 = -d$ etc.
Point C is the reflection of B in line OP. From $\arg p = \arg(1 - \bar{b}) = \frac{1}{2}(\pi - \varphi)$, we can get $\arg c = 2\arg p - \arg b = \pi - 2\varphi$, so $c = -\bar{b}^2$.
Now we can verify that EP bisects $\angle CED$. This happens if and only if $(p-e)^2(\bar{c} - \bar{e})(\bar{d} - \bar{e})$ is real. Since $\bar{d} - \bar{e} = -\sqrt{3}i$, this is equivalent with $\operatorname{Re}\left[(p-e)^2(\bar{c} - \bar{e})\right] = 0$. Here

$$\begin{aligned}
(p-e)^2(\bar{c} - \bar{e}) &= (1 - \bar{b} - e)^2(-b^2 - d) = (d - \bar{b})^2(-b^2 - d) \\
&= -|b|^4 + 2d|b|^2 b - d^2 b^2 - d\bar{b}^2 + 2d^2\bar{b} - d^3 \\
&= -1 - 2db + \bar{d}b^2 - d\bar{b}^2 - 2\overline{db} + 1 \\
&= -2(db - \overline{db}) + (\bar{d}b^2 - d\bar{b}^2),
\end{aligned}$$

whose real part is zero. It can be proved similarly that DP bisects $\angle EDC$. $\square$

Solutions to Problem 6

Answer. The only such m are $m = 2$ and $m = 10$.

Solution A. Consider an integer $m > 1$ for which the sequence defined in the problem statement contains only perfect squares. We shall first show that $m - 1$ is a power of 3.
Suppose that $m-1$ is even. Then $a_4 = 5m-1$ should be divisible by 4 and hence $m \equiv 1 \pmod 4$. But then $a_5 = 5m^2 + 3m - 1 \equiv 3 \pmod 4$ cannot be a square, a contradiction. Therefore $m - 1$ is odd.
Suppose that an odd prime $p \neq 3$ divides $m - 1$. Note that $a_n - a_{n-1} \equiv a_{n-2} - a_{n-3} \pmod p$. It follows that modulo p the sequence takes the form 1, 1, 4, 4, 7, 7, 10, 10, ...; indeed, a simple induction shows that $a_{2k} \equiv a_{2k-1} \equiv 3k - 2 \pmod p$ for $k \geq 1$. Since $\gcd(p, 3) = 1$ we get that the sequence $a_n \pmod p$ contains all the residues modulo p, a contradiction since only $(p+1)/2$ residues modulo p are squares. This shows that $m - 1$ is a power of 3.
Let h, k be integers such that $m = 3^k + 1$ and $a_4 = h^2$. We then have $5 \cdot 3^k = (h - 2)(h + 2)$. Since $\gcd(h - 2, h + 2) = 1$, it follows that $h - 2$ equals either $1, 3^k$ or 5, and $h + 2$ equals either $5 \cdot 3^k, 5$ or 3^k, respectively. In the first two cases we get $k = 0$ and in the last case we get $k = 2$. This implies that either $m = 2$ or $m = 10$.
We now show the converse. Suppose that $m = 2$ or $m = 10$. Let $t = 1$ or $t = 3$ so that $m = t^2+1$. Let $b_1, b_2, b_3, \ldots$ be a sequence of integers defined by $b_1 = 1, b_2 = 1, b_3 = 2$, and

$$b_n = tb_{n-1} + b_{n-2}\,, \quad \text{for all } n \geq 4\,.$$

Clearly, $a_n = b_n^2$ for $n = 1, 2, 3$. Note that if $m = 2$ then $a_4 = 9$ and $b_4 = 3$, and if $m = 10$ then $a_4 = 49$ and $b_4 = 7$. In both the cases we have $a_4 = b_4^2$.
If $n \geq 5$ then we have

$$b_n^2 + b_{n-3}^2 = (tb_{n-1} + b_{n-2})^2 + (b_{n-1} - tb_{n-2})^2 = (t^2 + 1)(b_{n-1}^2 + b_{n-2}^2) = m(b_{n-1}^2 + b_{n-2}^2)\,.$$

Therefore, it follows by induction that $a_n = b_n^2$ for all $n \geq 1$. This completes the solution. □

Solution B. We present an alternate proof that $m = 2$ and $m = 10$ are the only possible values of m with the required property.
Note that

$$\begin{aligned} a_4 &= 5m - 1\,, \\ a_5 &= 5m^2 + 3m - 1\,, \\ a_6 &= 5m^3 + 8m^2 - 2m - 4\,. \end{aligned}$$

Since a_4 and a_6 are squares, so is a_4a_6. We have

$$4a_4a_6 = 100m^4 + 140m^3 - 72m^2 - 72m + 16\,.$$

Notice that

$$\begin{aligned} (10m^2 + 7m - 7)^2 &= 100m^4 + 140m^3 - 91m^2 - 98m + 49 < 4a_4a_6\,, \\ (10m^2 + 7m - 5)^2 &= 100m^4 + 140m^3 - 51m^2 - 70m + 25 > 4a_4a_6\,, \end{aligned}$$

so we must have

$$4a_4a_6 = (10m^2 + 7m - 6)^2 = 100m^4 + 140m^3 - 71m^2 - 84m + 36\,.$$

This implies that $m^2 - 12m + 20 = 0$, so $m = 2$ or $m = 10$. □