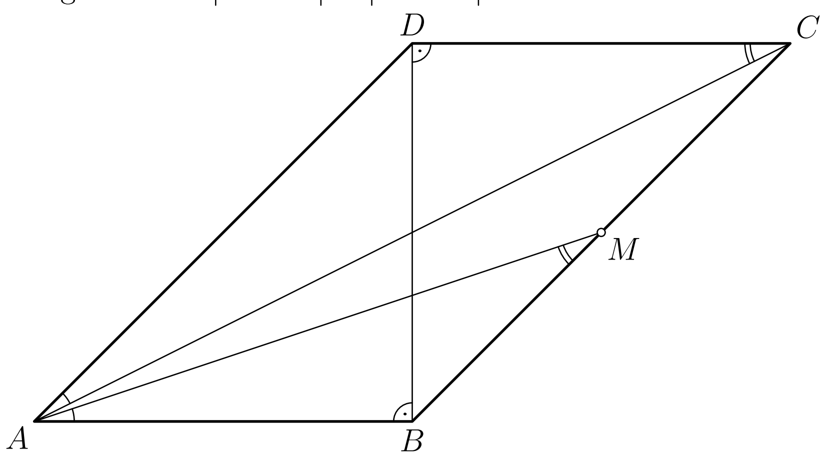
Solution 1. The ratio of the numbers on the board is invariant, so cannot change $1/2 \to 2/3$. Or the difference is semi-invariant (always increases), so cannot stay at 1.

Solution 2. For the lack of the primes the subset can not contain any number divisible by the prime greater then 3. Both 1 and 8 are third powers, they can be add to any convenient subset. The numbers in $P = \{2, 3, 4, 6, 9\}$ remains. If the number 6 is in the subset the only numbers 4 and 9 must follow it from the P. The convenient subsets of $P \setminus \{6\}$ are empty set, $\{2, 4\}$, $\{3, 9\}$ and its union. Thus only five subsets of P satisfy the condition which means that there are $5 \cdot 2(\text{number } 1) \cdot 2(\text{number } 8) - 1(\text{empty set}) = 19$ convenient subsets in total.

Solution 3. From the given condition it follows that ABD and CBD are two congruent right-angled isosceles triangles (SAS) and thus the quadrilateral $ABCD$ is the parallelogram with $|\angle ABC| = |\angle CDA| = 135°$.



If we suppose $|AB| = |BD| = |DC| = 1$ we obtain $|BC| = |AD| = \sqrt{2}$ and $|BM| = \frac{1}{2}\sqrt{2}$. Just we shall prove that the triangles ABM are ADC are similar because

$$|AM| : |BM| = 1 : \frac{\sqrt{2}}{2} = \sqrt{2} : 1 = |AD| : |DC|.$$

It directly follows

$$|\angle BAM| + |\angle DCA| = 45°.$$

Solution 4. If $abc \geq 0$ we have

$$ab + bc + ca \leq abc \leq 2abc$$

an the claim is true.

We will prove that the case $abc < 0$ is impossible. For the sake of contradiction let us suppose that it holds. Then just one from a, b, c is negative, wlog $c < 0$ (i.e., a and b are positive and $a + b > 0$ holds). Using $c = 1 - (a + b)$ we rewrite

$ab + bc + ca < abc$ by the following manner

$$ab + (1 - (a + b))(a + b) = ab + c(a + b) = ab + bc + ca < abc = ab(1 - (a + b)).$$

It means

$$(1 - (a + b))(a + b) < -ab(a + b), \quad \text{i.e.} \quad 1 - a - b + ab < 0,$$

or equivalently

$$(a - 1)(b - 1) < 0$$

what is the desired contradiction for the positive integers a and b.

Remark: We can prove that both $a + b + c = 1$ and $ab + bc + ca < abc$ give $ab + bc + ca \leq 0 \leq abc$.

Solution 5. The desired number n equals 14499

Adding algorithm gives that

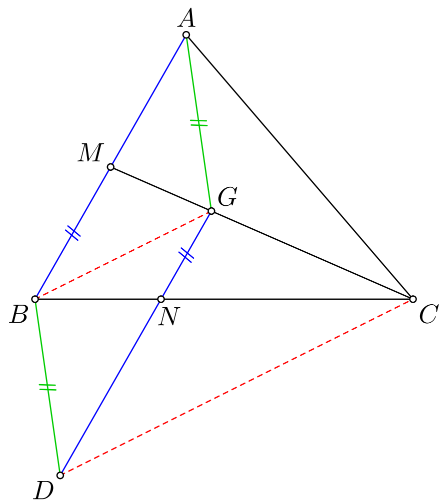
$$S(2n) = 2 * S(n) - 9k,$$

where k is a number of digits in n which are greater then 4. Using this we rewrite the equation to the form $4S(n) = 3(2S(n) - 9k)$, or equivalently $2S(n) = 27k$. It follows $27 \mid S(n) > 0$. Case $S(n) \geq 54$ means that n is written by at least 6 digits.

For $S(n) = 27$ the number n has just two digits greater than or equal to 5, remaining at least $(27 - 2 \cdot 9) : 4 \geq 3$ digits are at most 4. The least desired n consists of at least 5 digits, two of them are greater than 4. In the least n such digits are the greatest possible last two and the remaining three are the first. By easy argumentation we obtain $n = 14499$.

Solution 1. Let N is the meet point of BC and GD. From $|CG| : |CM| = 2 : 3$ it follows $|BN| = 1/3|BC|$ and $|GN| = 2/3|MB| = 1/3|AB|$. This yields $|CN| = 2/3|BC|$ and $|DN| = |DG| - 1/3|AB| = |AB| - 1/3|AB| = 2/3|AB|$. Triangles BNG and CND are homothetic with the centre N and coefficient -2 which follows the parallelism.

Other solution. Let S_{XYZ} be the area of a triangle XYZ. Then $S_{BCG} = 1/3\, S_{ABC} = S_{ABG} = S_{DBG}$ which follows the parallelism of BG and CD.



Solution 2. The triple satisfies

$$ab - 588c - 2024 = 14\sqrt{3}\,(ac - b).$$

Rationality of the lhs gives $b = ac$, i.e.

$$c(a^2 - 588) = 2024.$$

Maximal a means minimal positive integer c. For $c = 1$ it is $a^2 = 2612$ where a is not integer, for $c = 2$ it is $a^2 = 1600$ with $a = 40$. The desired triple is $(40, 80, 2)$.

Solution 3. Let an isosceles triangle ABC with interior angles $\alpha = \beta$, γ (in the natural order) satisfies the problem. According to the problem conditions it is sufficient to consider a cutting line going through the triangle vertex. We will show that only triangles with the interior angles $(45°, 45°, 90°)$, $(36°, 36°, 108°)$, $(72°, 72°, 36°)$ and $((540/7)°, (540/7)°, (180/7)°)$ satisfy the problem.

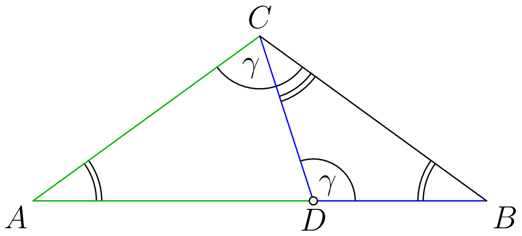
(1) $\gamma = 90°$ (right-angled isosceles triangle). The cutting line goes through the point C obviously, it is its altitude and it corresponds to the first solution.

(2) $\gamma > 90°$ (obtuse isosceles triangle). The cutting line through the point C (obviously) meets the point $D \in AB$ and wlog the angle CDB is obtuse. Then $|ADC| = \frac{1}{2}(180° - (90° - \frac{1}{2}\gamma)) = 45° + \frac{1}{4}\gamma$ and $|BDC| = \gamma$. This

follows

$$180° = |ADC| + |BDC| = (45° + \tfrac{1}{4}\gamma) + \gamma, \quad \text{a tedy} \quad \gamma = 108°,$$

which corresponds with the second solution.



(3) $\gamma < 90°$. The cutting line must go through A or B, wlog A, and meets the point $E \in BC$. We distinguish two cases according to AE is the base or the leg of the triangle ABE.

(a) AE is the leg. Then

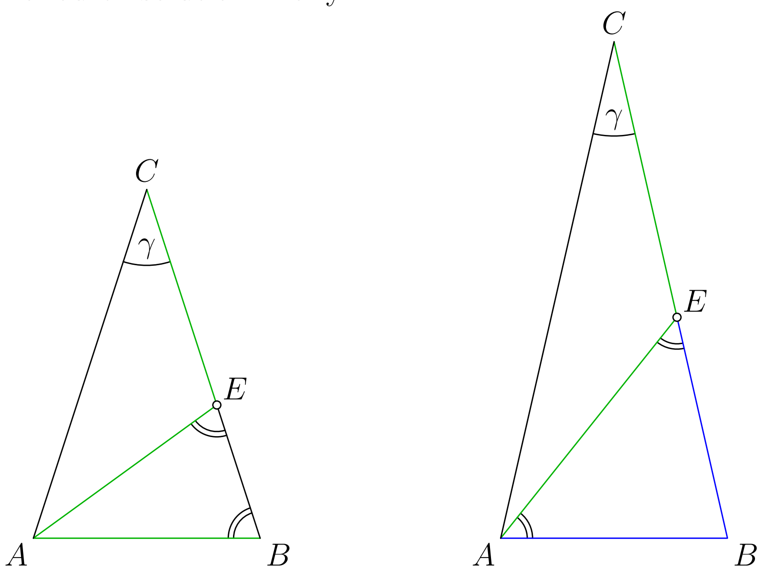
$$180° = |AEB| + |AEC| = (90° - \tfrac{1}{2}\gamma) + (180° - 2\gamma), \ \text{tedy} \ \gamma = 36°,$$

the third solution,

(b) AE is the base. Then

$$180° = |AEB| + |AEC| = (45° + \tfrac{1}{4}\gamma) + (180° - 2\gamma), \ \text{tj.} \ \gamma = 180°/7,$$

the fourth solution finally.



Solution 4. Let $\lfloor\sqrt{n}\rfloor = k$. The n could be written in the form $k^2 + z = (k-1)^2 + 2(k-1) + 1 + z$, where $0 \le z \le 2k$. The assumption $k - 1 \mid k^2 + z$ means $k - 1 \mid z + 1$.

- For $k = 1$ there is no solution.
- For $k = 2$ all permissible $z \in \langle 0, 4\rangle$ satisfy, i.e 5 solutions.

- For $k = 3$ only even $z \in \langle 0, 6 \rangle$ satisfies, i.e 3 solutions.
- For $k = 4$ only $z \in \{2, 5, 8\} \subset \langle 0, 8 \rangle$ satisfies, i.e three solutions.
- For $k > 4$ the inequality $3(k-1) > 2k+1 \geq z+1$ holds, so we have only 2 possibilities to satisfy $k-1 \mid z+1$:
  (1) $z+1 = k-1$, i.e. $z = k-2$. Since $k > 4$, $z \in \langle 0, 2k \rangle$ holds. Then $n = k^2 + k - 2$. The maximal k satisfying $n < 2024$ is 44, in this case we have $k = 5, 6, \ldots, 44$, i.e. 40 possibilities.
  (2) $z+1 = 2k-2$, i.e. $z = 2k-3$. analogically $z \in \langle 0, 2k \rangle$. Then $n = k^2 + 2k - 3$. As in the previous case we obtain $k = 5, 6, \ldots, 44$, i.e 40 possibilities once more.

Altogether we have $5 + 3 + 3 + 40 + 40 = 91$ suitable numbers n.

Solution 5. No. Both numbers have the same number of digits, this implies their ratio lies between 1/10 and 10, precisely 1/8 and 8. These two powers of 2 would need to be congruent modulo 9. Meanwhile their difference can only be $2^{a+1} - 2^a = 2^a$, $2^{a+2} - 2^a = 3 \cdot 2^a$ or $2^{a+3} - 2^a = 7 \cdot 2^a$. None of them is divisible by 9.

Solution 6. If there is 1 anywhere, then the complete row and complete column are filled with 1's, so 1's are everywhere. Now say there are no 1's on the board. Let n be the greatest number present on the board. Then together in the row and column containing n we need to have n distinct values. But only $n-1$ different values (i.e. 2 to n) are available (as we assumed there are no 1's) – contradiction.