CAPS Match 2024: Solutions

ISTA, Austria
June 30 – July 3, 2024

Problem 1. Determine whether there exist 2024 distinct positive integers satisfying the following: If we consider every possible ratio between two distinct numbers (we include both a/b and b/a), we will obtain numbers with finite decimal expansions (after the decimal point) of mutually distinct non-zero lengths. (Patrik Bak, Slovakia)

Solution. We will show these numbers exist. For that we define sequences $a_1, a_2, \ldots, a_{2024}$ and $b_1, b_2, \ldots, b_{2024}$ and then consider numbers $c_i = 2^{a_i} \cdot 5^{b_i}$ for $i = 1, 2, \ldots, 2024$.

We choose the sequences a_i and b_i in such a way that a_i is increasing, b_i is decreasing, and the differences $a_i - a_j$ and $b_j - b_i$ were all mutually distinct for all indices $i > j$. This will be enough because

$$\frac{c_i}{c_j} = \frac{2^{a_i} \cdot 5^{b_i}}{2^{a_j} \cdot 5^{b_j}} = \frac{2^{a_i - a_j}}{5^{b_j - b_i}},$$

this number has a decimal expansion of a length $b_j - b_i$, whereas analogously, $\frac{c_j}{c_i}$ has a length of $a_i - a_j$.

We now construct the needed sequences, starting with a_i. We will do it inductively. Take $a_1 = 1$, $a_2 = 2$. When we have the numbers $a_1, a_2, \ldots, a_i$, then by choosing $a_{i+1} = 2a_i$ we will achieve $a_{i+1} - a_i > a_i - a_1$, therefore all newly added differences will be higher than the previous ones.

We can construct b_i similarly, starting at the end by taking $b_{2024} = a_{2024}$, then $b_{2023} = 2b_{2024}$, and so on. Since $b_{2023} - b_{2024} = a_{2024}$, all the differences in b_i will be at least $b_{2023} - b_{2024} = a_{2024}$.

Remark: In our construction, $a_i = 2^{i-1}$ and $b_i = 2^{4049-i}$.

Problem 2. For a positive integer n, an *n-configuration* is a family of sets $\langle A_{i,j}\rangle_{1\le i,j\le n}$. An n-configuration is called *sweet* if for every pair of indices (i,j) with $1 \le i \le n-1$ and $1 \le j \le n$ we have $A_{i,j} \subseteq A_{i+1,j}$ and $A_{j,i} \subseteq A_{j,i+1}$. Let $f(n,k)$ denote the number of sweet n-configurations such that $A_{n,n} \subseteq \{1,2,\ldots,k\}$. Determine which number is larger: $f(2024, 2024^2)$ or $f(2024^2, 2024)$. (Wojciech Nadara, Poland)

Solution. Consider a sweet n-configuration $\langle A_{i,j}\rangle_{1\le i,j\le n}$ with $A_{n,n} \subset \{1,2,\ldots,k\}$. For any $x \in \{1,2,\ldots,k\}$ and $i \in \{1,2,\ldots,n\}$ define

$$p_x(i) = |\{j : x \in A_{i,j}\}|\,.$$

Since $A_{i,j} \subseteq A_{i,j+1}$ for all suitable i,j, the set $\{j : x \in A_{i,j}\}$ consists of $p_x(i)$ largest elements of $\{1,2,\ldots,n\}$. Since $A_{i,j} \subseteq A_{i+1,j}$ for all suitable i,j, the function $p_x\colon \{1,2,\ldots,n\} \to \{0,1,2,\ldots,n\}$ is nondecreasing. Therefore every sweet n-configuration determines a family $\langle p_x\rangle_{1\le x\le k}$ of nondecreasing functions $p_x\colon \{1,2,\ldots,n\} \to \{0,1,\ldots,n\}$. Conversely, every such a family determines a sweet n-configuration $\langle A_{i,j}\rangle_{1\le i,j\le n}$ with $A_{n,n} \subset \{1,2,\ldots,k\}$ in the following way: $A_{i,j} = \{x \in \{1,2,\ldots,k\} : j \ge n+1-p_x(i)\}$. Therefore $f(n,k) = g(n)^k$ where $g(n)$ is the number of nondecreasing functions $p\colon \{1,2,\ldots,n\} \to \{0,1,\ldots,n\}$.

Using the stars-and-bars method, there is a bijection between the family of nondecreasing functions $p\colon \{1,2,\ldots,n\} \to \{0,1,\ldots,n\}$ and the set of sequences consisting of n stars and n bars. The bijection is given by

$$p \longrightarrow \underbrace{**\ldots*}_{p(1)}|\underbrace{**\ldots*}_{p(2)-p(1)}|\underbrace{**\ldots*}_{p(3)-p(2)}|\ldots|\ \underbrace{**\ldots*}_{p(n)-p(n-1)}\ |\underbrace{**\ldots*}_{n-p(n)}$$

Thus $g(n) = \binom{2n}{n}$.

The problem boils down to determining which of the numbers

$$\binom{2n}{n}^{n^2}, \quad \binom{2n^2}{n^2}^{n},$$

where $n = 2024$, is larger. Note that

$$\binom{2n^2}{n^2} = \frac{\prod_{i=1}^{n^2}(n^2+i)}{\prod_{i=1}^{n^2} i} = \prod_{i=1}^{n^2}\left(\frac{n^2+i}{i}\right) = \prod_{j=0}^{n-1}\prod_{i=1}^{n}\frac{n^2+jn+i}{jn+i} > \prod_{j=0}^{n-1}\left(\frac{n^2+jn+n}{jn+n}\right)^n =$$

$$= \prod_{j=0}^{n-1}\left(\frac{n+j+1}{j+1}\right)^n = \left(\prod_{j=1}^{n}\frac{n+j}{j}\right)^n = \binom{2n}{n}^n$$

and therefore

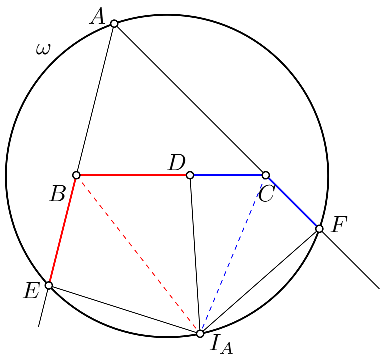
$$\binom{2n^2}{n^2}^n > \binom{2n}{n}^{n^2}.$$

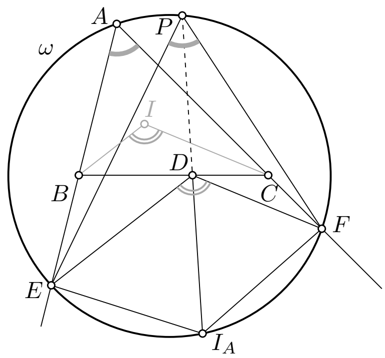
Remark: A sketch of a slightly different way of thinking about f(n,k)= $\binom{2n}{n}^k$: Consider an $n \times n$ table. In a cell with coordinates (i,j), list all the elements of the set $A_{i,j}$. Fix an element $x \in \{1,\ldots,k\}$ and consider the cells that contain the number x. By the condition, those cells form a region closed under making a step right and making a step up. Such regions are delimited by grid paths that start at $[0,n]$, end at $[n,0]$, and only steps right or down. There are $\binom{2n}{n}$ possible paths for each x, thus $f(n,k) = \binom{2n}{n}^k$.

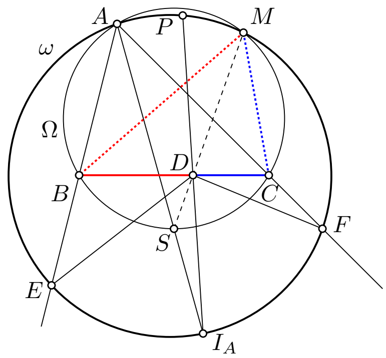
Problem 3. Let ABC be a triangle and D a point on its side BC. Points E, F lie on the lines AB, AC beyond vertices B, C, respectively, such that $BE = BD$ and $CF = CD$. Let P be a point such that D is the incenter of triangle PEF. Prove that P lies inside the circumcircle Ω of triangle ABC or on it. (Josef Tkadlec, Czech Republic)

Solution. Let ω the circumcircle of triangle AEF and let I_A be the A-excenter of triangle ABC. First, we prove that I_A is the midpoint of the arc EF of ω that does not contain point A (see the left figure).

To that end, note that since I_A lies on the external angle bisector of $\angle B$ and $BE = BD$, triangles BEI_A and BDI_A are congruent (SAS). Similarly, triangles CFI_A and CDI_A are congruent, so in particular $I_AE = I_AF$. Moreover, $\angle BEI_A + \angle CFI_A = \angle BDI_A + \angle CDI_A = 180$, hence the points A, E, I_A, F lie on a single circle in this order.







Next, we prove that P is the second intersection of I_AD and ω (see the middle figure). Let I be the incenter of triangle ABC. Then $ED \parallel BI$ and $DF \parallel IC$. Setting $\angle EAF = \alpha$, we get $\angle EDF = \angle BIC = 90 + \frac{1}{2}\alpha$, thus $\angle EPF = \alpha = \angle EAF$, so P lies on ω. Since I_A is the midpoint of arc, it lies on the angle bisector PD, so P lies both on I_AD and on ω as claimed.

Finally, we show that P lies on that arc of ω which lies inside Ω (see the right figure). Let $M \neq A$ be the second intersection of ω and Ω (if they are tangent, we set $M = A$). Then M is the center of the spiral similarity that maps BE to CF (alternatively, we angle-chase that triangles MBE and MCF are similar by AA). Thus $MB/MC = BE/CF = BD/DC$, so MD is the angle bisector of BMC, and so it passes through the midpoint S of the arc BC of Ω that does not contain A.

Now forget about points B, C, E, F and focus on circles Ω, ω and on the points A, M, I_A, S, D, P. Circles Ω and ω share points A and M. Being the A-excenter of ABC, point I_A belongs to that arc AM of ω which lies outside of Ω (e.g. since $AI_A > AS$). Point S lies on the segment AI_A and point D lies on the segment SM, so point D lies inside the angle AI_AM. Thus, point $P = I_AD \cap \omega$ belongs to the other arc AM of ω than I_A, namely to the one which lies inside Ω.

Problem 4. Let $ABCD$ be a quadrilateral, such that $AB = BC = CD$. There are points X, Y on rays CA, BD, respectively, such that $BX = CY$. Let P, Q, R, S be the midpoints of segments BX, CY, XD, YA, respectively. Prove that points P, Q, R, S lie on a circle. (Michal Pecho, Slovakia)

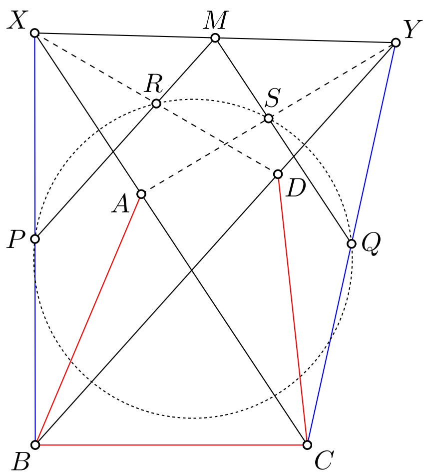
Solution. Let M be the midpoint of XY. Note that PR is midline in triangles XBD and XBY, hence M lies on PR. Analogously M lies on QS.

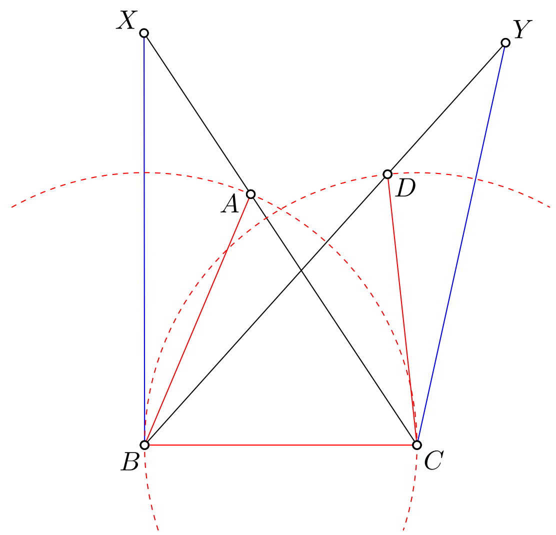
Let ω_1 be a circle with center B and radius $AB = BC$ and ω_2 be a circle with center C and radius $BC = CD$.

Distance of X from center of ω_1 is the same sa distance of Y from center of ω_2 and also ω_1 and ω_2 have radius of same size, hence power of X with respect to ω_1 is the same as power of Y with respect to ω_2, so

$$XA \cdot XC = YD \cdot YB.$$

Using homotheties centered at X, Y we get that $MS \cdot MQ = MR \cdot MP$ and thus points P, Q, R, S lie on a circle.





Problem 5. Let $\alpha \neq 0$ be a real number. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + y^2) = f(x-y)f(x+y) + \alpha y f(y)$$

holds for all x, $y \in \mathbb{R}$. (Walther Janous, Austria)

Solution. *Answer:* For every $\alpha \neq 0$, the zero function and the function with value 1 at 0, but 0 elsewhere are solutions. For $\alpha = 2$, the identity function $x \mapsto x$ is another solution.

Solution-check. The linear function clearly works. Consider the function f such that $f(0) = 1$ and $f(x) = 0$ otherwise. Note that $yf(y) = 0$ for all real numbers y. Then it is sufficient to realize that $f(x^2 + y^2) \neq 0$ iff $x = 0 \wedge y = 0$. Similarly $f(x-y)f(x+y) \neq 0$ iff $x + y = x - y = 0 \Leftrightarrow x = 0 \wedge y = 0$ which shows that also this function is a solution.

Proof. Denote by $P(x, y)$ the proposition in the problem statement. Comparing $P(x, y)$ with $P(x, -y)$ yields $f(y) = -f(-y)$ for all $y \neq 0$. Using this equality, $P(y, x)$ shows that $2f(x-y)f(x+y) = \alpha(xf(x) - yf(y))$ for $x \neq y$. Plugging this into the original equation, we obtain

$$f(x^2 + y^2) = \frac{\alpha}{2}(xf(x) + yf(y))$$

for $x \neq y$. Setting $y = 0$ in this equation shows $f(x^2) = \alpha x f(x)/2$ for $x \neq 0$, whereas $P(x, 0)$ gives $f(x^2) = f(x)^2$ for all $x \in \mathbb{R}$. Hence $\alpha x f(x)/2 = f(x)^2$, that is, $f(x) = 0$ or $f(x) = \alpha x/2$ for $x \neq 0$. In particular, if $f(x) \neq 0$, then $f(x) = \alpha x/2$. On the other hand, $P(0, 0)$ shows $f(0) = f(0)^2$ and therefore $f(0) = 0$ or $f(0) = 1$.

Consider first the case that $f(x) = 0$ for all $x \neq 0$. Then both possible values for $f(0)$ yield functions fulfilling the original equation (if $(x, y) \neq (0, 0)$, all terms in $P(x, y)$ are zero anyway and $(x, y) = (0, 0)$ was treated before).

Now for the other case: There is a real number $z \neq 0$ satisfying $f(z) = \alpha z/2$. Then $f(z^2) = f(z)^2 = (\alpha/2)^2 z^2 \neq 0$, and hence $f(z^2) = (\alpha/2)z^2$. By comparing the last two statements, we obtain $\alpha = 2$ and then $f(z) = z$.

- $f(0) = 1$. Consider $P(z/2, z/2)$: $f(z^2/2) = z + zf(z/2)$. The left-hand side is 0 or $z^2/2$, the right-hand side z or $z + z^2/2$. Since $z \neq 0$, only $z^2/2 = z \iff z = 2$ and $0 = z + z^2/2 \iff z = -2$ are possible. Either way, $f(2) = 2$ and $f(-2) = -2$, because f is odd. But then $f(4) = f(2^2) = f(2)^2 = 4$, which is impossible, because we just proved that $z = 2$ and $z = -2$ are the only real numbers with $f(z) = z$.
- $f(0) = 0$. We show that $f(x) = 0$ for all positive reals x if f is not the identity function:
  1. There are $0 < a < b$ with $f(a) = 0$, $f(b) = b$. Then $P(x, y)$ for $x = \sqrt{b-a}$ and $y = \sqrt{a}$ yields
     $$0 \neq b = f(b) = f(x-y)f(x+y) + 2f(a) = f(x-y)f(x+y),$$
     hence $f(x-y) = x - y$ and $f(x+y) = x + y$ and $b = x^2 - y^2 = b - 2a$, forcing the contradiction $a = 0$.
  2. There are $0 < a < b$ with $f(a) = a$, $f(b) = 0$. Analogous to Case 1, we arrive at the contradiction $b = 0$ when investigating $P(\sqrt{b-a}, \sqrt{a})$.

  Except for the identity, we only have $f(x) = 0$ for $x > 0$ and thus $f(x) = 0$ for $x \neq 0$ as possible solution, which we have already found and treated before.

Problem 6. Determine whether there exist infinitely many triples (a, b, c) of positive integers such that p divides $\lfloor (a + b\sqrt{2024})^p \rfloor - c$ for every prime p.

Note: $\lfloor x \rfloor$ denotes the largest integer not larger than x. (Walther Janous, Austria)

Solution.

Let $D := 2024$. Consider any pair of positive integers (a, b) such that $0 < a - b\sqrt{D} < 1$. One can easily find an infinite number of such pairs by choosing $a = \lceil b\sqrt{D} \rceil$. Then

$$(a + b\sqrt{D})^p + (a - b\sqrt{D})^p = 2a^p + 2\sum_{k=1}^{\infty} \binom{p}{2k} a^{p-2k} b^{2k} D^k \in \mathbb{Z}$$

is larger than $(a+b\sqrt{D})^p$, since we add a positive term, but it is smaller than $(a+b\sqrt{D})^p+1$. As it is integer and $p \mid \binom{p}{2k}$ for all $1 \le k \le \frac{p-1}{2}$, we see that

$$\begin{aligned} \left\lfloor (a + b\sqrt{D})^p \right\rfloor &= (a + b\sqrt{D})^p + (a - b\sqrt{D})^p - 1 \\ &= 2a^p + 2\sum_{k=1}^{\infty} \binom{p}{2k} a^{p-2k} b^{2k} D^k - 1 \equiv 2a - 1 \pmod{p} \end{aligned}$$

by Fermat's little theorem. Observe that this congruence is also valid for $p = 2$, although $2 \nmid \binom{2}{2 \cdot 1}$, because the sum is taken twice anyway. Therefore, choosing $c := 2a - 1$, we get $p \mid \left\lfloor (a + b\sqrt{D})^p \right\rfloor - c$ for all primes p.

In summary, for any positive integer b we get a triple $(\lceil sqrt2024b \rceil, b, 2\lceil sqrt2024b \rceil - 1)$ that has the desired property.