18TH MEMO
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Solution Booklet

The Problem Selection Committee

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gratefully received

132 problem proposals submitted by 8 countries:

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Contents

I-1

Determine all $k \in \mathbb{N}_0$ for which there exists a function $f\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(2024) = k$ and

$$f(f(n)) \le f(n+1) - f(n)$$

for all $n \in \mathbb{N}_0$.

Remark. Here $\mathbb{N}_0$ denotes the set of nonnegative integers.

Answer. The possible values of $f(2024)$ are $0, 1, 2, \ldots, 2023$.

Solution. Note that $0 \le f(f(n)) \le f(n+1) - f(n)$, hence f is increasing.

Claim. *$f(n) \le n$ for all $n \in \mathbb{N}_0$.*

Proof. Suppose indirectly that $f(n) > n$, i.e., $f(n) \ge n+1$. By monotonicity, this implies $f(f(n)) \ge f(n+1)$. Consequently,

$$f(n+1) \le f(f(n)) \le f(n+1) - f(n),$$

leading to $f(n) \le 0$, which contradicts $0 \le n < f(n)$. □

The claim immediately yields $f(2024) \le 2024$. However, $f(2024) = 2024$ is impossible as it would mean $f(f(2024)) = f(2024) = 2024 \le f(2025) - f(2024) = f(2025) - 2024$ or $4048 \le f(2025)$, contradicting the claim for $n = 2025$.

On the other hand, for any $0 \le k \le 2023$ the function

$$f(n) = \begin{cases} 0 & n \le 2023 \\ k & n \ge 2024 \end{cases}$$

satisfies the condition, as $f(0) = f(k) = 0$, hence $f(f(n)) = 0$ for all $n \in \mathbb{N}_0$.

Solution 2. We only give a new proof for the claim.

Suppose indirectly that $f(n) > n$. Using the condition of the problem

$$\begin{aligned} f(f(n)) &\leq f(n+1) - f(n) \\ f(f(n+1)) &\leq f(n+2) - f(n+1) \\ f(f(n+2)) &\leq f(n+3) - f(n+2) \\ &\vdots \\ f(f(f(n)-1)) &\leq f(f(n)) - f(f(n)-1) \end{aligned}$$

Summing up the inequalities, we get

$$f(f(n)) \leq f(f(n)) + f(f(n+1)) + f(f(n+2)) + \cdots + f(f(f(n)-1)) \leq f(f(n)) - f(n),$$

leading to $f(n) \leq 0$, which contradicts $0 \leq n < f(n)$.

I-2

There is a sheet of paper (like this one) on an infinite blackboard. Marvin secretly chooses a convex 2024-gon P that lies fully on the piece of paper. Tigerin wants to find the vertices of P. In each step, Tigerin can draw a line g on the blackboard that is fully outside the piece of paper, then Marvin replies with the line h parallel to g that is the closest to g which passes through at least one vertex of P. Prove that there exists a positive integer n such that Tigerin can always determine the vertices of P in at most n steps.

Solution 1. One of the key observations is the following. If 3 answer lines intersect at a common point X, then X must be a vertex of P.

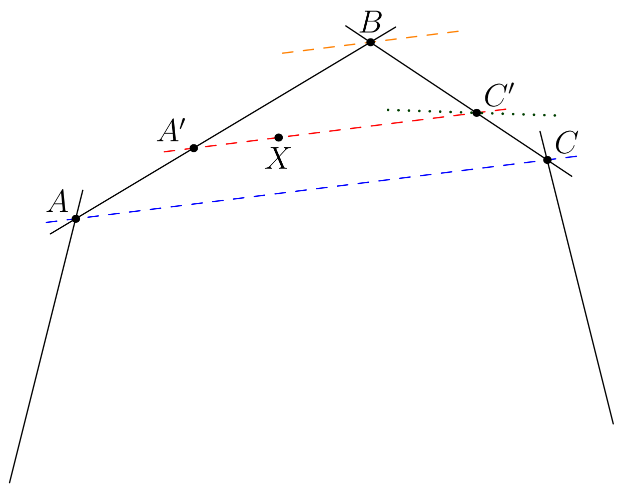
Let us start by querying the four sides of the paper. This determines a rectangle on the paper which contains P and each of the sides of the rectangle contain at least one vertex of P.

Let us assume that Q is the convex polygon given by the intersection of the closed half-planes with the answered line boundaries obtained so far containing P. Let's call a vertex of Q good if at least 3 answered lines have passed through it already, and call it bad otherwise.

If we have not found all vertices of P yet, i.e. there are less than 2024 good vertices, then there must be still at least one bad vertex of Q, since P is inside Q and if all vertices of Q were good, P couldn't have more vertices than Q.

Now suppose that we have not found all vertices of P yet. Then we repeat the following until we have not found all vertices of P yet. Pick B, a bad vertex of Q, and let A and C be its neighbours on Q, and query for a line parallel with AC outside the paper in half-plane AC containing B. We claim that repeatedly querying this way results in us finding all vertices of P in a bounded number of steps.

Let us look at the different cases based on which vertex (or possibly 2 vertices) of P the answer to this query contains.



Case 1: The answer passes through B (orange). Then B is a new good vertex. Therefore this case can happen at most 2024 times.

Case 2: The answer is the line AC itself (blue). Then A and C are points of P. We might or might not have known this before, but now we additionally know that they are direct neighbors on P. Therefore this case can happen at most 2024 times as we didn't know they were direct neighbours before this query.

Case 3: At least one vertex X of P on the answered line is in the interior of Q. In this case the answer intersects AB in A' and BC in C' (red line on the figure above). Once a vertex of P is on the boundary of Q, it can never become an interior point of Q again, therefore this case can happen at most 2024 times as X was not on the boundary of Q before this query.

Case 4: The vertex (or vertices) of P on the answered line is (are) in the line segment AB or BC, excluding endpoints. In this case the answer intersects AB in A' and BC in C' (red line on the figure above) and so either A' or C' is a vertex of P. Once a vertex of P becomes a vertex of Q, it can never become a non-vertex of Q again, therefore this case can happen at most 2024 times as A' and C' weren't vertices of Q before this query.

To summarize, our algorithm is as follows: Pick a bad vertex B. Query for a line parallel to AC. If we get a line passing through B (orange) or through AC (blue), there is nothing else to do. Otherwise, pick a new bad B and repeat. All cases can only occur at most 2024 times each, therefore the algorithm stops and all vertices of Q are good at this point. Then $P = Q$.

Note: When applying the algorithm we do not know if the current answer is case 3 or case 4. This solution proves that $n = 4 \cdot 2025$ suffices.

Solution 2. We present an alternative argument that cases 3 and 4 from the above solution can happen finitely many times.

Note that in both cases the number of the sides of Q is increased by one.

We can observe that the maximum number of sides of Q is at most 4048, because each side of Q contains a vertex of P and each vertex of P can be part of at most two sides of Q. This means that cases 3 or 4 can occur at most 4048 times in a row, as otherwise the number of sides of Q would increase by at least 4049 times in a row. Thus before the first case 1 or case 2, and between any case 1 and case 2, and after the last case 1 or case 2, at most 4048 steps can be taken. Thus the algorithm stops in at most $4048 \cdot (2024 + 2024 + 1)$ steps and we found $P = Q$ as above.

Solution 3. We use the same observation as the previous solution. If 3 answer lines intersect at a common point X, then X must be a vertex of P. Start by querying the 4 sides of the paper. Define Q, good vertices and bad vertices as above. In this solution we use the pigeonhole principle with the observation to find vertices of P.

We will prove that we can always find a new vertex in at most $2024 \cdot 3 + 2$ steps if we haven't found all vertices of P yet. Hence we find P in at most $n = 4 + 2024 \cdot (2024 \cdot 3 + 2)$ steps. (Note that we could give a much better n with a bit more care.)

Assume that we have found $k < 2024$ vertices of P so far. By design, P lies in Q and both are convex, so if all of Q's vertices were good, $P = Q$ and we are done. Hence there is at least one bad vertex of Q.

In case there are two bad neighboring vertices A and B of Q, and let C be the next vertex of Q after A and B in this order. As the line AB is a side of Q, it must be an answered line, so it contains a vertex of P. Furthermore, in this line, only the points of segment AB are contained in Q, hence P has a vertex on segment AB. Choose an arbitrary point C' inside the segment BC, and let us query a line parallel to AC' outside the paper in half-plane AC' containing B. It is easy to see that the answer we get must intersect segment AB (possibly going through one of its endpoints), and it cannot pass through any other vertex of Q. Therefore it cannot pass through a previously known good vertex. Hence with $2024 \cdot 2 + 1$ different such queries (always choosing a different point C' from BC), by the pigeonhole principle there must be a vertex of P with at least 3 of these answered lines passing through it, and so we found a new good vertex. So in this case, we find a new vertex in at most $2024 \cdot 2 + 1$ many queries.

If there are no neighboring vertices of Q which are both bad, then there must be neighboring vertices A, B, C with A and C being good, and B being bad. Let us query a line parallel to AC outside the paper in half-plane AC containing B. We know that the boundary line of the answer intersects Q, and the given half-plane contains A and C, hence there are 3 options.

If the boundary line goes through B, then it is a new good vertex as this is the third answer line going through it. In this case we immediately found a new good vertex, so in this case, we find a new vertex in 1 query.

If the boundary line doesn't go through B, and also doesn't go through A and C, then it must intersect the segments AB and BC at some points A' and C'. Then A' and C' become neighboring bad vertices of the polygon obtained by the intersection of Q with this half-plane, and so we can apply the previous case to find a new vertex of P in at most $2024 \cdot 2 + 1$ queries. So in this case, we find a new vertex in at most $2024 \cdot 2 + 2$ many queries.

Finally, if the boundary line is AC, then the intersection of Q and this half-plane has one less side. This case can happen at most 2024 times, as in this case we find a side of P. Hence, after at most 2024 steps we can apply one of the previous cases finding a new good vertex to find a new vertex of P. So in this case, we find a new vertex in at most $2024 \cdot 3 + 2$ many queries. This finishes the proof.

Solution 4. Let $m = 2024 \cdot 2 + 1$. Then, by the pigeonhole principle, if we query m pairwise non-parallel lines, there is a vertex of P through which at least 3 of the m lines pass through. Call a vertex of P found if we have already received at least 3 answer lines passing through it.

We start by asking any m pairwise non-parallel lines outside the paper and asking m lines parallel to the first m lines so that the paper is in the strips for each pair of parallel lines. Then there is a vertex A_1 of P through which at least 3 answered lines go. However, of each parallel line pair, at most one can go through A_1, so at least m answered lines do not go through A_1. Then there is a vertex $A_2 \neq A_1$ of P through which at least 3 of the answered lines go through. So we know that A_1 and A_2 are vertices of P.

Now suppose we already have found vertices $A_1, A_2, \ldots, A_k$ of P forming convex polygon P' so that they are in this order on the boundary of P', and suppose that $k < 2024$. We will show that we can find a new vertex of P in a bounded number of queries.

We first query a line parallel to A_1A_2 outside the paper so that P' and the queried line fall on different sides of line A_1A_2. This tells us either that A_1A_2 is edge of P, or gives us a line ℓ parallel to A_1A_2 containing a vertex of P, which we have not found yet.

When we get the line ℓ, we choose m different points $(X_i)_{1\leq i\leq m}$ on the perpendicular bisector of A_1A_2 such that they lie in between ℓ and A_1A_2 and all the points of P' lie on the same side of lines A_1X_i and A_2X_i for all i. We query m lines parallel to A_1X_i outside the paper so that they are closer to A_1 than A_2 and m lines outside the paper parallel to X_iA_2 so that they are closer to A_2 than A_1.

Notice that from each pair of answered lines (parallel to A_1X_i and X_iA_2 respectively) at least one must not pass through any found vertex, as the only vertex of P the answers can go through are A_1 and A_2, so otherwise line ℓ could not touch P. Thus there are at least m of the $2m$ answered lines not passing through any vertex of P', hence we find a new vertex.

By repeating the steps above, the first case of finding a side of P can happen at most 2024 times, and if it does not happen, we find a new vertex in at most $1 + 2m$ queries. Therefore we find P in at most $n = m + m + 2024 + 2022 \cdot (1 + 2m)$ many steps.

I-3

Let ABC be an acute scalene triangle. Choose a circle ω passing through B and C which intersects segments AB and AC again in points $D \neq A$ and $E \neq A$, respectively. Let F be the intersection of BE and CD. Let G be the point on the circumcircle of ABF such that GB is tangent to ω. Similarly, let H be the point on the circumcircle of ACF such that HC is tangent to ω. Prove that there exists a point $T \neq A$, independent of the choice of ω, such that the circumcircle of AGH passes through T.

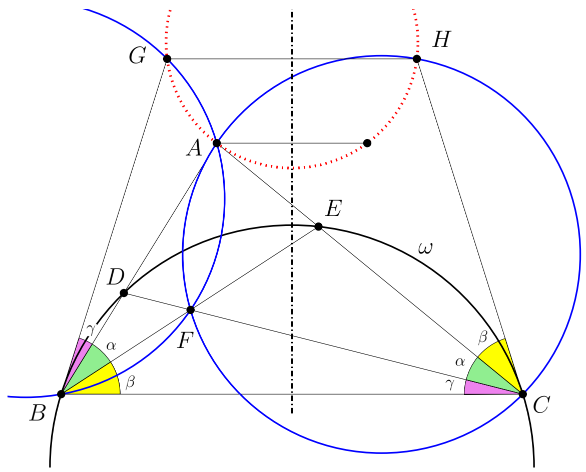
Solution. We will prove that BG and CH are symmetric in the perpendicular bisector of BC.

First, we will show $\angle CBG = \angle HCB$. Denote the angles $\angle EBD = \angle ECD = \alpha$, $\angle CBF = \beta$, $\angle FCB = \gamma$. Due to the tangency, $\angle ABG = \angle DCB = \gamma$ and $\angle HCA = \angle CBE = \beta$. From this we can see that both angles $\angle CBG$ and $\angle HCB$ are equal to $\alpha + \beta + \gamma$.

Now, we will prove that $BG = CH$, which will be enough to show the symmetry. Firstly, notice that due to $\angle FBA = \angle ACF$, the circumcircles of ABF and ACF have the same radii. We will prove that $\angle GAB = 180° - \angle CAH$ and that will be enough. We will do it by angle chasing:

$$\begin{aligned}\angle GAB + \angle CAH = (180° - \angle BGA - \angle ABG) + (180° - \angle AHC - \angle HCA) = \\ (\angle AFB - \gamma) + (\angle CFA - \beta) = (\angle AFB + \angle CFA) - (\beta + \gamma) = \\ (360° - \angle BFC) - (180° - \angle BFC) = 180°.\end{aligned}$$

Finally, the claim is now obvious due to symmetry: Let T be the reflection of A in the perpendicular bisector of BC. Then, A, T, G, H are clearly concyclic due to symmetry and T is independent from D and E, therefore the solution is complete.



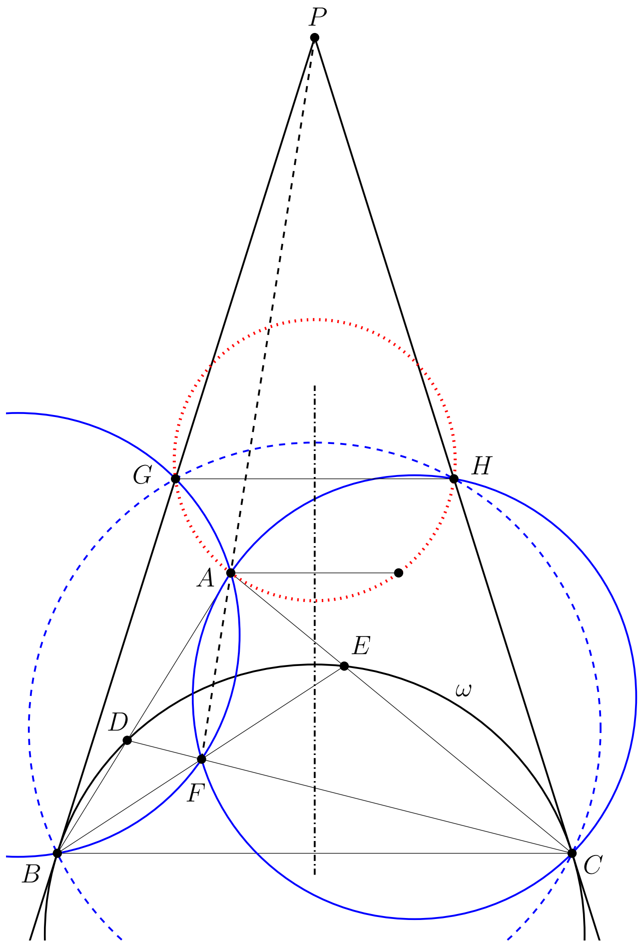
Solution 2. We will work on the projective plane.

Claim. *Lines BG, AF, CH are concurrent.*

Proof. Apply Pascal's theorem on the hexagon $BBECCD$ inscribed in ω: as $BE \cap CD = F$ and $BD \cap CE = A$, we get these two points with the intersection of the tangents at B and C are collinear. □

Denote the point of concurrence by P. By the power of point, $PG \cdot PB = PA \cdot PF = PH \cdot PC$, so we conclude that $BCHG$ is cyclic. Since lines BG, CH are symmetric with respect to the perpendicular bisector of BC, we have that $BCHG$ is an isosceles trapezoid. (If P happens to be a point at infinity, we have that quadrilaterals $BFAG$ and $CFAH$ are isosceles trapezoids, which can only happen if $BCHG$ is also an isosceles trapezoid.)

We finish the problem by noticing that the circumcircle of AGH is symmetric with respect to the bisector of BC, thus it must pass through T, the reflection of A in the perpendicular bisector of BC, which is invariant of D and E.



Alternative proof of the claim Notice that the intersection of lines BG, CH is the pole of line BC (with respect to ω). By La Hire's theorem, it suffices to show that the pole of line AF lies on BC. Now consider the cyclic quadrilateral $BECD$. Note that $BE \cap CD = F$ and $EC \cap DB = A$. Brocard's theorem tells us that the pole of AF is the intersection of diagonals BC, DE. This clearly proves our claim.

Solution 3. (based on the solution of Grzegorz Kaczmarek) This solution uses isogonal conjugacy. We will use the fact that in a quadrilateral $ABCD$ a point P has an isogonal conjugate if and only if $\angle APB + \angle CPD = 180°$. For a detailed proof, see the comment below.

Similarly to the other solutions, one can prove that $\angle BAG + \angle CAH = 180°$, which is the same as $\angle BFG + \angle CFH = 180°$. It follows that in quadrilateral $BCHG$ point F has an isogonal conjugate. Denote this by F'. We claim that it is the desired point T.

Firstly, as point F' has an isogonal conjugate in $BCHG$, we have that $\angle BF'C + \angle HF'G = 180°$. Note that $\angle BF'C = 180° - \angle F'BC - \angle F'CB = 180° - \angle FBG - \angle FCH = 180° - (180° - \angle FAG) - (180° - \angle FAH) = 180° - \angle HAG$. Thus, $\angle HAG = \angle HF'G$, meaning that points A, G, H, F' are, in fact, concyclic.

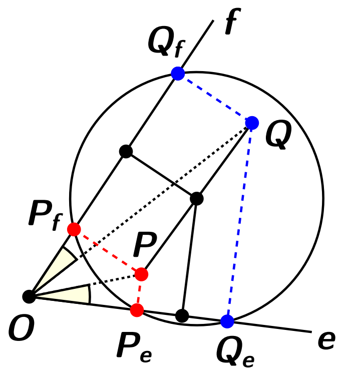
It remains to show that point F' is fixed, or in other words, it is independent of the choice of ω. Luckily, this is fairly easy: since GB is tangent to ω, $\angle F'BC = \angle FBG = \angle EBG = \angle ECB = \angle ACB$, so line BF' is fix. Similarly, line CF' is fix too, therefore F' must be fix, as well. The proof is complete.

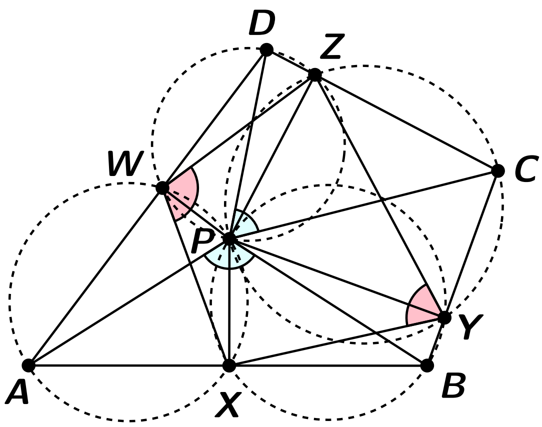
Comment. Let's discuss isogonal conjugates in order to prove the used fact.

Lemma. Suppose we have two lines, e and f intersecting at O. Let P and Q be two different points (not lying on any of the lines) such that OP and OQ are isogonal with respect to e and f (or in other words, lines OP and OQ are symmetric with respect to an angle bisector of lines e, f). Then, the projections of P and Q in lines e and f all lie on a circle with center being the midpoint of PQ.

Proof. Denote the projections of P and Q onto lines e and f by P_e, P_f, Q_e, Q_f, respectively. Note that quadrilaterals PP_eOP_f, QQ_fOQ_e are similar, as you can obtain one from the other by a reflection in the angle bisector of lines e, f followed by a suitable homothety with center O. It follows (from the reflection) that P_eP_f and Q_eQ_f are antiparallel, so $P_eP_fQ_fQ_e$ is cyclic. The center of the circle is the intersection of the bisectors of segments P_eQ_e, P_fQ_f. These bisectors are the midlines of the right-angled trapezoids PP_eQ_eQ, PP_fQ_fQ. Hence, the center must be the midpoint of PQ. □

We use this lemma to show the following property of isogonal conjugates in polygons: a point P has an isogonal conjugate if and only if its projections onto the sides are concyclic. Indeed, if the point has an isogonal conjugate P^*, then for any two adjacent sides the projections of the two points lie on a circle with center being the midpoint PP^*, thus all projections must lie on the same circle. The other direction follows similarly but in reverse: the reflection of P in the center of the circe through the projections must be the isogonal conjugate (the proof of the lemma also works in reverse).





Now we are ready to prove our fact. Denote the projections of P onto sides AB, BC, CD, DA by X, Y, Z, W, respectively. Due to the right angles, we have that quadrilaterals $AXPW$, $BYPX$, $CZPY$, $DWPZ$ are cyclic. Note that $\angle APB + \angle CPD = (\angle APX + \angle XPB) + (\angle CPZ + \angle ZPD) = \angle AWX + \angle XYB + \angle CYZ + \angle ZWD = 360^\circ - \angle XYZ - \angle ZWX$. Therefore, we have

$$\angle APB + \angle CPD = 180^\circ \iff \angle XYZ + \angle ZWX = 180^\circ \iff XYZW \text{ is cyclic.}$$

Hence, $\angle APB + \angle CPD = 180^\circ$ is equivalent to P having an isogonal conjugate.

I-4

For any positive integer n, let $\sigma(n)$ denote the sum of positive divisors of n. Determine all polynomials P with integer coefficients such that $P(k)$ is divisible by $\sigma(k)$ for all positive integers k.

Solution 1. We are going to use the following well-known lemma:

Lemma. *For any integers a, b and polynomial p with integer coefficients we have*

$$a - b \mid p(a) - p(b).$$

Let $p \neq q$ be any prime numbers. Then from $\sigma(pq) = (p+1)(q+1)$ we have

$$(p+1)(q+1) \mid P(pq).$$

This is equivalent to

$$pq - (-p - q - 1) \mid P(pq).$$

Using this and the lemma for $a = pq$ and $b = -p - q - 1$ we get that

$$(p+1)(q+1) \mid P(-p - q - 1).$$

From this $-p - 1 \mid P(-p - q - 1)$ also follows. Now using the lemma for $a = -p - q - 1$ and $b = -q$ we get that $-p - 1 \mid P(-p - q - 1) - P(-q)$. Consequently

$$p + 1 \mid P(-q)$$

for any primes p and q. But this implies that $P(-q) = 0$ for every prime q. Therefore P has infinitely many roots, thus it is the zero polynomial.

Solution 2. Let p, q be primes, then $\sigma(pq) = (1+p)(1+q)$, so $(1+p)(1+q) \mid P(pq)$. Let $P(x) = \sum_{k=0}^{\deg P} a_k x^k$. Using the fact that $pq \equiv -p \pmod{1+q}$, we can conclude the following:

$$0 \equiv P(pq) \equiv \sum_{k=0}^{\deg P} a_k (pq)^k \equiv \sum_{k=0}^{\deg P} a_k (-p)^k \equiv P(-p) \qquad \pmod{1+q}.$$

So we have that $1 + q \mid P(-p)$ for any p, q prime.

As p, q were arbitrary, we can vary q and obtain $P(-p) = 0$, and as p was arbitrary, we get that $P(-p) = 0$ for all primes p, so $P(x) = 0$ for all x as desired.

Solution 3. Let us assume that there is such a polynomial P different from the zero polynomial. Then it has some fixed degree, let us denote it by c.

Let p be any prime. From $\sigma(p^k) = 1 + p + \ldots + p^k$ we have that

$$1 + p + \ldots + p^k \mid P(p^k).$$

Let us define a new polynomial P_k as follows: $P_k(x) := P(x^k)$. We know that $1 + p + \ldots + p^k \mid P_k(p^k)$ for every prime p. But since there are infinitely many primes, and the polynomial $1 + x + \ldots + x^k$ has 1 as its main coefficient, it has to divide P_k as a polynomial.

Let us consider any number l, and let d be any divisor of it. Then using the above conclusion for $k = \frac{l}{d}$ and plugging x^d into x we have

$$1 + x^d + x^{2d} + \ldots + x^l \mid P(x^l)$$

(again, as polynomials.)

Let us recall the well-known fact that the greatest common divisor of $x^m - 1$ and $x^n - 1$ (as polynomials) is $x^{\gcd(m,n)} - 1$. We claim that if d_1 and d_2 are divisors of l and $l + d_1$ and $l + d_2$ are coprime, then $1 + x^{d_1} + x^{2d_1} + \ldots + x^l$ and $1 + x^{d_2} + x^{2d_2} + \ldots + x^l$ are coprime too. Indeed the former divides $x^{l+d_1} - 1$ and the later divides $x^{l+d_2} - 1$, and they are not divisible by $x - 1$.

Now let us take l such that it has divisors $d_1, d_2, \ldots, d_{c+1}$ with the property that $l + d_i$ and $l + d_j$ are coprime for any $i \neq j$. (It is easy to find such an l.) Then the polynomials $1 + x^{d_i} + x^{2d_i} + \ldots + x^l$ are all pairwisely coprime, thus their product divides $P(x^l)$. On one hand from the fact that P has degree c we know that $P(x^l)$ has degree lc. On the other hand it is divisible by the product $\prod_{i=1}^{c+1}(1 + x^{d_i} + x^{2d_i} + \ldots + x^l)$, which has degree $l(c + 1)$, which is a contradiction.

Comment. An alternate ending of Solution 3 from the fact that $Q_k(x) = 1 + x + \ldots x^k | P(x^k)$ for all k is the following.

For all k, the first primitive k-th root of unity ($z_{k+1} = \cos\left(\frac{2\pi}{k+1}\right) + i \sin\left(\frac{2\pi}{k+1}\right)$) is a root of Q_k. Therefore $P(z_{k+1}^k) = 0$, thus $u_k = z_{k+1}^k = \cos\left(\frac{2k\pi}{k+1}\right) + i \sin\left(\frac{2k\pi}{k+1}\right)$ is a root of P. But if $l \neq k$, then $u_k \neq u_l$, therefore P has infinitely many roots, meaning that P is the zero polynomial.

Solution 4. Let us assume that P is not identically 0, $P(x) = x^{k+1}Q(x) + ax^k$ (here ax^k is the term with the lowest exponent). Let $p \nmid a$ be a prime, and $q_1, q_2, ..., q_{k+1}$ be different primes such that all of them are congruent to -1 modulo p (it is possible to take such primes from the Dirichlet theorem). Now let $N = pq_1q_2 \ldots q_{k+1}$. By the fact that σ is multiplicative we have $\sigma(N) = \sigma(q_1) \cdot \sigma(q_2) \cdot \ldots \sigma(q_n) \cdot \sigma(p) = (q_1 + 1) \cdot (q_2 + 1) \cdot \ldots (q_n + 1) \cdot (p + 1)$, which is divisible by p^{k+1}, thus $p^{k+1}|\sigma(N)|P(N)$, because σ. On the other hand p^{k+1} does not divide $P(N)$, since $P(N) = N^{k+1}Q(N) + aN^k$, and $p^{k+1} \nmid aN^k$, which is a contradiction.

Solution 5. Let p be any prime number. Choose n, such that $p \mid n$, while $p^2 \nmid n$. Then by the fact that σ is multiplicative, we get $p+1 \mid \sigma(n)$ and thus $p+1 \mid P(n)$. By the Chinese Remainder Theorem we can get $n_1, n_2 \dots n_{p+1}$, such that these have all different residues modulo $p+1$, and $p \mid n_i$, while $p^2 \nmid n_i$, thus in particular $p+1 \mid P(n_i)$ for all $1 \leq i \leq p+1$. Consider any natural number k, then there is i such that $n_i \equiv k \pmod{p+1}$. Thus $P(k) \equiv P(n_i) \pmod{p+1}$, therefore $p+1$ divides $P(k)$ for every integer k. Since p was any prime we get that $P \equiv 0$.

Solution 6. First fix a prime p, and substitute p^α into the polynomial P. Then we have

$$\sigma(p^\alpha) = \frac{p^{\alpha+1}-1}{p-1}|P(p^\alpha).$$

So for infinitely many natural numbers n, we have $\frac{pn-1}{p-1}|P(n)$. For rational numbers $r, s \in \mathbb{Q}$, we denote by $r|s$ if $\frac{s}{r} \in \mathbb{Z}$. We will need the following lemma.

Lemma. *Given two polynomials $Q, R \in \mathbb{Q}[x]$ such that for infinitely many $n \in \mathbb{Z}$ we have $Q(n)|R(n)$, then $Q(x)|R(x)$ in $\mathbb{Q}[x]$.*

Proof. Since $\mathbb{Q}[x]$ is a Euclidian domain, $R(x)$ can be written as follows $R(x) = Q(x)S(x) + T(x)$, where $\deg T \leq \deg Q$. There is a positive integer N, such that $R'(x) := N^2 \cdot R(x), Q'(x) := N \cdot Q(x), S'(x) := N \cdot S(x), T'(x) := N^2 \cdot T(x)$ have all integer coefficients. Now the same equation holds for the modified polynomials: $R'(x) = Q'(x)S'(x) + T'(x)$. Also the property that for infinitely many n we have $Q'(n)|R'(n)$ remains true. Combining these we get that $Q'(n)|T'(n)$ for infinitely many n. But T' has smaller degree than Q', so for large enough n, $|T'(n)| < |Q'(T)|$, therefore $T' \equiv 0$. □

Applying the lemma we get that for all prime p we have $\frac{px-1}{p-1}|P(x)$, so P has infinitely many roots, thus it is the zero polynomial.

T-1

Consider the two infinite sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ of real numbers such that $a_0 = 0$, $b_0 = 0$ and

$$a_{k+1} = b_k, \qquad b_{k+1} = \frac{a_k b_k + a_k + 1}{b_k + 1}$$

for each integer $k \geq 0$. Prove that $a_{2024} + b_{2024} \geq 88$.

Solution. Let us notice that a_i is the same as the sequence b_i shifted by one. So the whole problem might be reduced to the sequence of b_i. The definition of b_{k+1} can be rewritten as:

$$b_{k+1} = \frac{a_k b_k + a_k + 1}{b_k + 1} = a_k + \frac{1}{b_k + 1} = b_{k-1} + \frac{1}{b_k + 1}.$$

Now if we define a new sequence $B_i = b_i + 1$, then we arrive at

$$B_{k+1} = B_{k-1} + \frac{1}{B_k},$$

multiplying with B_k gives

$$B_k B_{k+1} = B_{k-1} B_k + 1.$$

Hence, using $C_k = B_k B_{k+1}$ we can see that for any $k \geq 1$,

$$C_k = C_{k-1} + 1.$$

Since $C_0 = 1 \cdot 2 = 2$, we obtain $C_{2023} = 2025$. Now we just have to remember that

$$C_{2023} = B_{2023} B_{2024} = (b_{2023} + 1)(b_{2024} + 1) = (a_{2024} + 1)(b_{2024} + 1).$$

Using the AM-GM inequality,

$$45 = \sqrt{2025} = \sqrt{(a_{2024} + 1)(b_{2024} + 1)} \leq \frac{(a_{2024} + 1) + (b_{2024} + 1)}{2},$$

which gives $88 \leq a_{2024} + b_{2024}$, as we seeked to prove.

Solution 2. As in Solution 1, we substitute $B_k = b_k + 1$ and get $B_{k+1} = B_{k-1} + \frac{1}{B_k}$. Then by calculating the first few terms, we conjecture that

$$B_k = \begin{cases} \dfrac{(k+1)\cdot(k-1)\cdot(k-3)\cdot\ldots\cdot 2}{k\cdot(k-2)\cdot(k-4)\cdot\ldots\cdot 1}, & \text{if } k \text{ is odd} \\[2ex] \dfrac{(k+1)\cdot(k-1)\cdot(k-3)\cdot\ldots\cdot 1}{k\cdot(k-2)\cdot(k-4)\cdot\ldots\cdot 2}, & \text{if } k \text{ is even,} \end{cases}$$

which is easily proven by induction. From this we can see that

$$B_{2024} = \frac{2025}{B_{2023}}.$$

The statement we want to prove is equivalent to:

$$B_{2023} + \frac{2025}{B_{2023}} \geq 90 \iff B_{2023}^2 - 90\cdot B_{2023} + 2025 \geq 0 \iff (B_{2023} - 45)^2 \geq 0,$$

which clearly holds.

T-2

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$yf(x+1) = f(x+y-f(x)) + f(x)f(f(y))$$

for all $x, y \in \mathbb{R}$.

Answer. The functions $f(x) = x$ for all $x \in \mathbb{R}$ and $f(x) = 0$ for all $x \in \mathbb{R}$ are the only solutions.

Solution.

Claim. *$f(x) = cx + d$ is a solution if and only if $d = 0$ and $c = 0$ or $c = 1$.*

Proof. It is easy to check that $f(x) = 0$ and $f(x) = x$ are indeed solutions. There is no more constant solutions, hence we assume that $c \neq 0$ for the rest of the proof.

Plugging in $f(x) = cx + d$ into the functional equation gives

$$y(c(x+1)+d) = c(x+y-cx-d) + d + (cx+d)(c(cy+d)+d).$$

This equation is true for all $x, y \in \mathbb{R}$ if and only if all the coefficients of the (multivariate) polynomials equal. Hence, considering the coefficient of xy gives $c = c^3$, so $c = 1$ or $c = -1$.

The coefficient of x gives $0 = c - c^2 + c^2 d + cd$, hence $0 = 1 - c + d + cd$. If $c = 1$ then $d = 0$, and if $c = -1$ then this gives a contradiction, finishing the proof of the claim. $\square$

Assume that $f(t_0) = 0$ for some $t_0 \in \mathbb{R}$. Plugging in (t_0, y) gives

$$yf(t_0+1) = f(t_0+y).$$

It follows that f is linear, hence we are done by the Claim. From now we assume that $f(x) \neq 0$ for all $x \in \mathbb{R}$.

Plugging in $(x, f(x)+1)$ gives

$$f(x)f(x+1) + f(x+1) = f(x+1) + f(x)f(f(f(x)+1)),$$

hence

$$f(x+1) = f(f(f(x)+1)). \tag{1}$$

Claim. *f is injective.*

Proof. Assume that $f(a) = f(b)$ for some $a \neq b$. It follows from (1) that

$$f(a+1) = f(f(f(a)+1)) = f(f(f(b)+1)) = f(b+1).$$

Plugging in (a, b) and (b, a) into the original equation and using $f(a) = f(b)$ gives

$$bf(a+1) = f(a+b-f(a)) + f(a)f(f(b)) = f(b+a-f(b)) + f(b)f(f(a)) = af(b+1),$$

hence $a = b$, as $f(a+1) = f(b+1) \neq 0$. $\square$

By (1) and injectivity we get $x + 1 = f(f(x) + 1)$. Plugging $x = -1$ gives $f(f(-1) + 1) = 0$ which contradicts the assumption that $f(x) \neq 0$ for all $x \in \mathbb{R}$. This finishes the proof.

Solution 2. Throughout the solution f^n denotes the n-th iterate of f, i.e., $f^1(x) = f(x)$ and $f^n(x) = f(f^{n-1}(x))$ for all $x \in \mathbb{R}$ and $n > 1$. We only show a second proof for the statement that if $f(x) \neq 0$ for all $x \in \mathbb{R}$ then there is no solution.

Assume that $f(x) \neq 0$ for all $x \in \mathbb{R}$. Plugging in $(x, f(x))$ gives

$$f(x+1) = 1 + f^3(x)$$

for all x (upon dividing by $f(x) \neq 0$) whereas $(x, f(x) + 1)$ gives

$$f(x+1) = f(f(1+f(x))) = f(1+f^4(x)) = 1 + f^7(x).$$

which gives in particular that $f^3(x) = f^7(x)$. We can now compute:

$$\begin{aligned} f(x+2) &= 1 + f^3(1+x) \\ &= 1 + f^2(1+f^3(x)) \\ &= 1 + f(1+f^6(x)) \\ &= 2 + f^9(x) \\ &= 2 + f^5(x) \end{aligned}$$

and similarly we have

$$\begin{aligned} f(x+3) &= 2 + f^5(1+x) \\ &= 2 + f^4(1 + f^3(x)) \\ &= 2 + f^3(1 + f^6(x)) \\ &= 2 + f^2(1 + f^9(x)) \\ &= 2 + f(1 + f^{12}(x)) \\ &= 3 + f^{15}(x) \\ &= 3 + f^{11}(x) \\ &= 3 + f^7(x) \\ &= 2 + f(x+1) \end{aligned}$$

and we obtain $f(x+2) = f(x) + 2$ for all $x \in \mathbb{R}$. If we now plug in $(x+2, y)$ we have

$$yf(x+1) + 2y = f(x + y - f(x)) + f(x)f(f(y)) + 2f(f(y))$$

which gives $f(f(y)) = y$ so $f(f(0)) = 0$, a contradiction.

Solution 3. We reduce to showing that $f(x) = 0$ for some $x \in \mathbb{R}$, and reduce this to injectivity, as in Solution 1. Therefore we assume $f(x) \neq 0$ for all $x \in \mathbb{R}$, but $f(x_1) = f(x_2) = a$ for some $x_1 \neq x_2$. Comparing the given equation for (x_1, y) and (x_2, y) gives

$$y(f(x_1+1) - f(x_2+1)) = f(x_1 + y - a) - f(x_2 + y - a).$$

For $y = a \neq 0$ we get $f(x_1 + 1) = f(x_2 + 1)$, so the left side above is identically 0. Therefore

$$f(x_1 + y - a) = f(x_2 + y - a)$$

for all y, so f is periodic with period $t = x_1 - x_2 \neq 0$.

But now comparing the original equation for (x, y) and $(x, y+t)$ for any real x, y we get

$$tf(x+1) = 0,$$

so f attains the value 0.

T-3

There are 2024 mathematicians sitting in a row next to the river Tisza. Each of them is working on exactly one research topic, and if two mathematicians are working on the same topic, everyone sitting between them is also working on it.

Marvin is trying to figure out for each pair of mathematicians whether they are working on the same topic. He is allowed to ask each mathematician the following question: "How many of these 2024 mathematicians are working on your topic?" He asks the questions one by one, so he knows all previous answers before he asks the next one.

Determine the smallest positive integer k such that Marvin can always accomplish his goal with at most k questions.

Answer. The number of required questions is 2023.

Solution. We solve the problem more generally, for n mathematicians. We will prove that the answer is $n-1$.

By asking the left-most $n-1$ mathematicians, Marvin can determine the working groups from the left to right.

Now we show that $n-2$ questions may not be enough. Let x_i be the answer of the i-th mathematician. It is easy to see that $x_1^{-1}+\cdots+x_n^{-1}$ is the number of different topics studied. All of the mathematicians will answer 1 or 2 to the question in such a way, that after each question if a and b are the smallest and largest indices such that the a-th and b-th mathematicians haven't been asked by Marvin yet, then $x_1^{-1}+\cdots+x_{a-1}^{-1}$ and $x_{b+1}^{-1}+\cdots+x_n^{-1}$ are integers, and all values x_i with $a<i<b$ that have been already asked satisfy $x_i=2$. Suppose that Marvin asked x_k just now, and he asked at most $n-3$ questions before this, and by induction the constraint above is satisfied.

- If there are i, j such that $1 \leq i<k<j \leq n$ such that x_i and x_j haven't been asked, the answer will be $x_k=2$, and it is easy to check that all constraints are still satisfied.
- If for all $1 \leq i<k$ Marvin has already asked x_i: Let $m>k$ be the largest integer such that we already asked x_j for all $k<j \leq m$. This m exists and is less than n since he asked at most $n-3$ questions before this. Therefore, for all $k<j \leq m$, $x_j=2$. By the induction hypothesis, $x_1^{-1}+\cdots+x_{k-1}^{-1}$ is an integer, so the constraint can be satisfied by answering $x_k=2$ if $m-k$ is odd and $x_k=1$ if $m-k$ is even.
- If for all $k<i \leq n$ Marvin has already asked x_i: The answer can be decided similarly as in the previous case.

Therefore after $n-2$ questions we have the following: Only the a-th and b-th mathematicians weren't asked by Marvin, everyone between them answered 2, and everyone not sitting between them has a research topic different from the a-th and b-th mathematician (the topics can be reconstructed similarly as in the argument showing that $n-1$ questions are always enough).

If $b-a$ is odd, then $x_a = 1$ and $x_b = 1$ can be reconstructed, and $x_a = 2$ and $x_b = 2$ can be reconstructed as well, so Marvin can't decide which mathematicians share the same topic.

If $b-a$ is even, then $x_a = 1$ and $x_b = 2$ can be reconstructed, and $x_a = 2$ and $x_b = 1$ can be reconstructed as well, so again Marvin can't decide which mathematicians share the same topic.

T-4

A finite sequence $x_1, x_2, \ldots, x_r$ of positive integers is a *palindrome* if $x_i = x_{r+1-i}$ for all integers $1 \leq i \leq r$.

Let $a_1, a_2, \ldots$ be an infinite sequence of positive integers. For a positive integer $j \geq 2$, denote by $a[j]$ the finite subsequence $a_1, a_2, \ldots, a_{j-1}$. Suppose that there exists a strictly increasing infinite sequence $b_1, b_2, \ldots$ of positive integers such that for every positive integer n, the subsequence $a[b_n]$ is a palindrome and $b_{n+2} \leq b_{n+1} + b_n$. Prove that there exists a positive integer T such that $a_i = a_{i+T}$ for every positive integer i.

Solution. Define a *break point* to be a positive integer k such that $a[k]$ is a palidrome. Let $c_1 < c_2 < \ldots$ be a strictly increasing sequence of *all* break points. Then $c_{n+2} \leq c_{n+1} + c_n$ also holds whenever $c_{n+2} > b_2$. Namely, if $b_{j-1} < c_{n+2} \leq b_j$, then $c_{n+1} \geq b_{j-1}$ and $c_n \geq b_{j-2}$.

For positive integers p and q, let $p \sim q$ denote the fact that $a_p = a_q$.
Let x, $x+y$ and $x+y+z$ be three consecutive break points greater than b_2. From the condition, we have $z \leq x$. Consider any positive integer $r < x$.

Since x is a break point, $r \sim x - r$.
Since $x + y$ is a break point, $x - r \sim x + y - (x - r) = y + r$.
Since $x + y + z$ is a break point, $y + r \sim x + z - r$.

Hence, $r \sim x + z - r$ for all $r < x$. This implies that $x + z$ is also a break point, which means $y = z$ since we considered consecutive break points. Repeating this argument for the next three break points, we can conclude that there exists an arithmetic sequence of break points with common difference z.

Let $(x + nz)_n$ be an arithmetic sequence of break points. Consider any positive integer r. Let n be a positive integer such that $x + nz > r$.

Since $x + nz$ is a break point, $r \sim x + nz - r$.
Since $x + (n+1)z$ is a break point, $x + nz - r \sim x + (n+1)z - (x + nz - r) = z + r$.

Hence, the sequence $(a_n)_n$ is periodic with period z, which proves the claim.

Solution 2. Similarly to the first solution, $x \sim y$ denotes that $a_x = a_y$. If we know that $a[b]$ is palindromic, then $x \sim b - x$ for all $0 < x < b$. The notation $a[x, y]$ denotes $(a_x, a_{x+1}, ..., a_{y-1})$. Throughout the proof, all variables denote integers.

We show that the sequence is periodic with $b_2 - b_1$. Note that to show this, it suffices to show that $a[b_n]$ is periodic with $b_2 - b_1$ for all n. So the problem statement follows from the following proposition.

Main proposition. $a[b_{n+k}]$ is periodic with $b_{n+1} - b_n$ for all $n, k \geq 1$.

We prove the proposition by induction on k.

The case $k = 1$. If $0 < x < b_n$ then by palindromeness of $a[b_n]$ and $a[b_{n+1}]$, we have $x \sim b_n - x \sim b_{n+1} - (b_n - x) = x + (b_{n+1} - b_n)$, so $a[b_{n+1}]$ is periodic with $b_{n+1} - b_n$.

The case $k = 2$. We prove the following helpful lemma:

Lemma. Suppose that $t \geq 1$ and $0 < x < y$. If $a[x]$ is periodic with t, and $a[y]$ is palindromic, and $2x - y \geq t$, then $a[y]$ is periodic with t too.

Proof of lemma. Since $a[x]$ is periodic with t, by palindromeness of $a[y]$ we have that $a[y - x, y]$ is periodic with t too. If the two segments $a[x]$ and $a[y - x, y]$ overlap in at least t elements, this means that all pairs of distance t in $a[y]$ will be contained in at least one of the segments, so $a[y]$ is also periodic with t. Here, the overlap is $x - (y - x) = 2x - y \geq t$ indeed.

To use the lemma here, pick $t = b_{n+1} - b_n$, $x = b_{n+1}$ and $y = b_{n+2}$. We already know that $a[b_{n+1}]$ is periodic with t (by the $k = 1$ case), and we have $2x - y \geq t \Leftrightarrow b_{n+1} + b_n \geq b_{n+2}$ indeed, concluding the $k = 2$ case.

The case $k \geq 3$. To prove this case, we show that if $a[b_{n+k-1}]$ is periodic with $b_{n+1} - b_n$ and $a[b_{n+k}]$ is periodic with $b_{n+2} - b_{n+1}$ (both true by the inductive hypothesis) then $a[b_{n+k}]$ is periodic with $b_{n+1} - b_n$ too.

Let $t = b_{n+1} - b_n$ and $\Delta = b_{n+2} - b_{n+1}$.

Then we can inductively show that $a[b_{n+k-1} + \ell\Delta]$ is periodic with t for all $\ell \geq 0$. For $\ell = 0$ this is true by the $k = 1$ case. If for $\ell \geq 1$, we know that $a[b_{n+k-1} + (\ell - 1)\Delta]$ is periodic with t, then by the periodicity of $a[b_{n+k}]$ with Δ, we have that $a[\Delta, b_{n+k-1} + \ell\Delta]$ is also periodic with t. Also, the two t-periodic intervals overlap in at least t elements, as $b_{n+k-1} + (\ell - 2)\Delta \geq b_{n+k-1} - \Delta = b_{n+k-1} - b_{n+2} + b_{n+1} \geq b_{n+1} - b_n \Leftrightarrow b_{n+k-1} + b_n \geq b_{n+2}$, true since from $k \geq 3$, $b_{n+k-1} \geq b_{n+2}$. So $a[b_{n+k-1} + \ell\Delta]$ is also periodic with t.

Now choose ℓ so that $d - \Delta \leq b_{n+k-1} + \ell\Delta \leq b_{n+k}$, giving that we have some $0 \leq u \leq \Delta$ with $a[b_{n+k} - u]$ being t-periodic. Now by palindromeness of $a[b_{n+k}]$, also $a[u, b_{n+k}]$ is t-periodic, and the two intervals' overlap is at least t: $b_{n+k} - 2u \geq t \Leftrightarrow 2u \leq b_{n+k} - b_{n+1} + b_n$, and $2u \leq 2\Delta = 2b_{n+2} - 2b_{n+1} \leq b_{n+k} - b_{n+1} + b_n \Leftrightarrow 2b_{n+2} \leq b_{n+k} + b_{n+1} + b_n$, true since $b_{n+2} \leq b_{n+k}$ and $b_{n+2} \leq b_{n+1} + b_n$. So $a[b_{n+k}]$ is t-periodic, finishing the proof.

Solution 3. We will show that the sequence is periodic with $b_k - b_{k-1}$ where $k \geq 2$ is so that $b_k - b_{k-1} = \min_{1 \leq n} b_n - b_{n-1}$. First, we prove that b_k is periodic with $b_k - b_{k-1}$:

$$x \sim b_{k-1} - x \sim b_k - b_{k-1} + x \quad \forall x < b_{k-1},$$

where we have used that b_{k-1} and b_k are palindromic.

We will use the lemma from the above solution. Let's recall it.

Lemma. *Suppose that $t \geq 1$ and $0 < x < y$. If $a[x]$ is periodic with t, and $a[y]$ is palindromic, and $2x - y \geq t$, then $a[y]$ is periodic with t too.*

Let us apply the lemma repeatedly so that $t = b_k - b_{k-1}$, $x = b_{n-1}$ and $y = b_n$ for $n = k+1,\ k+2,\ k+3, \ldots$. The two conditions in the first sentence of the lemma are clearly satisfied at such applications, as well as that $a[y]$ is palindromic.

The condition $2x - y \geq t$ is equivalent to $2b_{n-1} - b_n \geq b_k - b_{k-1}$. Since $n \geq k \geq 2$, we know that $b_{n-1} + b_{n-2} \geq b_n$. Therefore $b_{n-1} - b_n \geq -b_{n-2}$. But since we chose k so that $b_k - b_{k-1}$ is smallest, we have that $2x - y \geq t$ is also satisfied at each application of the lemma.

The only remaining condition is that $a[x] = a[b_{n-1}]$ is periodic with period t. However, since the lemma states that $a[y] = a[b_n]$ is periodic with period t, at each application of the lemma we get that this condition is also satisfied for the next application. Therefore we get that $a[b_n]$ is periodic with period $b_k - b_{k-1}$ $\forall n \geq k$. This finishes the proof.

Solution 4. This solution uses the following lemma.

Lemma. *If a palindromic, finite sequence $(a_n)_{n=1}^{n=l}$ of length l is periodic with periods x and y with $x < y$ and $x + y \leq l + 1$, it is periodic with period $y - x$.*

Proof. For $r \leq x - 1$, we have $r + y \leq l$ and $x < r + y$, so by periodicity with y and x, we have $a_r = a_{r+y} = a_{r+y-x}$.

For $x < r \leq l - (y - x)$, we have $1 \leq r - x \leq l - y$, so by the periodicities we have $a_r = a_{r-x} = a_{r-x+y}$.

It remains to be proven that $a_x = a_y$. In case $x + y \leq l$, the first of the two above arguments works for $r = x$ and shows $a_x = a_y$, proving the lemma.

However, in case $x + y \nleq l$, by the condition of the lemma we have $x + y = l + 1$. Then by the palindromic condition, $a_x = a_{l+1-x} = a_y$. This finishes the proof of the lemma. □

Comment. Via Euler's algoritm, we may use the lemma repeatedly to give that the sequence is periodic with $\gcd(x, y)$.

As in the previous solution, we use that for all $n \geq 2$, $a[b_n]$ is periodic with period $b_n - b_{n-1}$. Then as $a[b_n]$ is a subsequence for $a[b_{n+1}]$, we have that $a[b_n]$ is also periodic with period $b_{n+1} - b_n$.

Now let's apply the lemma's remark for $a[b_n]$, which we know is palindromic. The two periods are $b_n - b_{n-1}$ and $b_{n+1} - b_n$. The condition $x + y \leq l + 1$ translates to $b_{n+1} - b_{n-1} \leq b_n$ (noting that $l = b_n - 1$). However, this is satisfied by the problem statement, so we get that $a[b_n]$ is periodic with period $\gcd(b_n - b_{n-1}, b_{n+1} - b_n)$.

Recalling that $a[b_{n+1}]$ is periodic with period $b_{n+1} - b_n$, which is a multiple of the period we got for $a[b_n]$. Since $b_{n+1} - b_n \leq b_{n-1} \leq b_n - 1$, an entire larger period is contained in $a[b_n]$.Then each larger period in $a[b_{n+1}]$ consists of smaller periods of length $\gcd(b_n - b_{n-1}, b_{n+1} - b_n)$ from $a[b_n]$, so $a[b_{n+1}]$ is also periodic with period $\gcd(b_n - b_{n-1}, b_{n+1} - b_n)$.

Then substituting $n + 1$ by n, $a[b_n]$ is periodic with period $\gcd(b_{n-1} - b_{n-2}, b_n - b_{n-1})$. So by the remark, it is periodic with period $\gcd(b_{n-1} - b_{n-2}, b_n - b_{n-1}, b_{n+1} - b_n)$. By repeatedly using the multiple-period argument, substituting $n + 1$ by n and using the remark, we get that $a[b_n]$ is periodic with $\gcd(b_2 - b_1, b_3 - b_2, \ldots, b_{n+1} - b_n)$.

Since $\big(\gcd(b_2 - b_1, b_3 - b_2, \ldots, b_{n+1} - b_n)\big)_n$ is a strictly decreasing positive integer sequence, it has a minimum which it achieves at some $n = k$. Then for $p = \gcd(b_2 - b_1, b_3 - b_2, \ldots, b_{k+1} - b_k)$, we have that $a[b_n]$ is periodic with period p for all $n \geq k$. This finishes the proof.

Comment. The lemma is true even if the palindromic condition is dropped. This stronger formulation requires a more in-depth, harder proof for the case $x + y = l + 1$, presented below.

Proof. We repeatedly perform the following moves, starting from index x. If the current index is at most $l - x$, we increase it by x. If we cannot perform this move, and the index is at least $y + 1$, we decrease it by y. If we cannot perform either move, we stop.

If at some point we cannot perform more steps, the index r we have satisfies $l - x < r < y + 1$, so $r = y$. Since at each step the elements of $(a_n)_{n=1}^{n=l}$ at the old and new index are equal due to periodicity, the elements at the first and last indices in our steps are equal, so $a_x = a_y$.

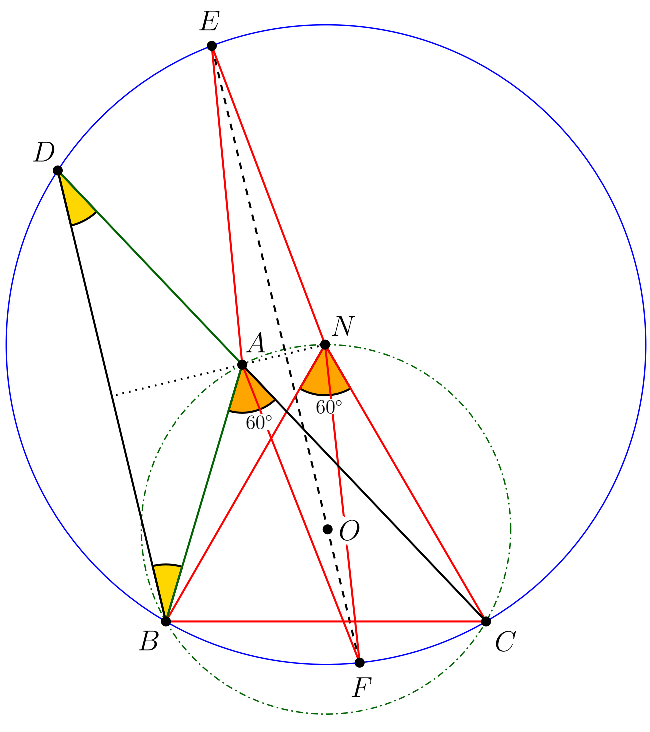
If we can perform the above steps infinitely many times, there will be an index at which we arrive at least twice. Let r_1 be the first such index. Then let the subsequent indices we get by the above steps from r_1 be $(r_n)_{n\geq 0}$. Since $r_1 = r_{k+1}$ for some index as we arrive at r_1 twice, we know that taking a step from r_k takes it to r_1, i.e. $r_1 = r_k + x$ or $r_1 = r_k - y$. However, at any index it is clear that we can only arrive by one type of moves, since then $1 \leq r - x \leq l - x$ and $l - x < r + y \leq l$ are both satisfied by r so $1 + x \leq r \leq l - y$, contradiction as $l = x + y - 1$. So the index preceding r_1 is also repeated at r_k as they cannot be different. So necessarily all indices repeat. But the index preceding the second time we reach x cannot be $0 = x - x$ and also cannot be $x + y = l + 1$, contradiction. So we can never take infinitely many such steps. □

T-5

Let ABC be a triangle with $\angle BAC = 60°$. Let D be a point on the line AC such that $AB = AD$ and A lies between C and D. Suppose that there are two points $E \neq F$ on the circumcircle of the triangle DBC such that $AE = AF = BC$. Prove that the line EF passes through the circumcenter of ABC.

Solution. Let N be the midpoint of arc BAC. Then triangle NBC is equilateral as $\angle BNC = 60°$ and N lies on the perpendicular bisector of BC. Moreover, N lies on the angle bisector of the angle DAB, which is the perpendicular bisector of segment BD considering the isosceles triangle ABD. Therefore, N is the circumcenter of triangle DBC.

Since NBC is equilateral, the circumradius of triangle DBC is the length of BC. So, we have $BC = NE = NF$ and we know that $AE = AF = BC$, thus $AENF$ is a rhombus. Hence, the line EF is the perpendicular bisector of AN. However, AN is a chord of the circumcircle of triangle ABC, so its perpendicular bisector passes through O. The solution is complete.



Solution 2. Let k be the circumcircle of triangle DBC and let K be its center. Denote by ω the circle with center A and radius BC. By definition, E and F are the two intersections of k and ω. Thus, we need to prove that O lies on the radical axis of the two circles.

Since $\angle BAC = 60°$, we have $\angle DAB = 120°$, and since $AB = AD$, we get $\angle BDA = \angle BDC =$

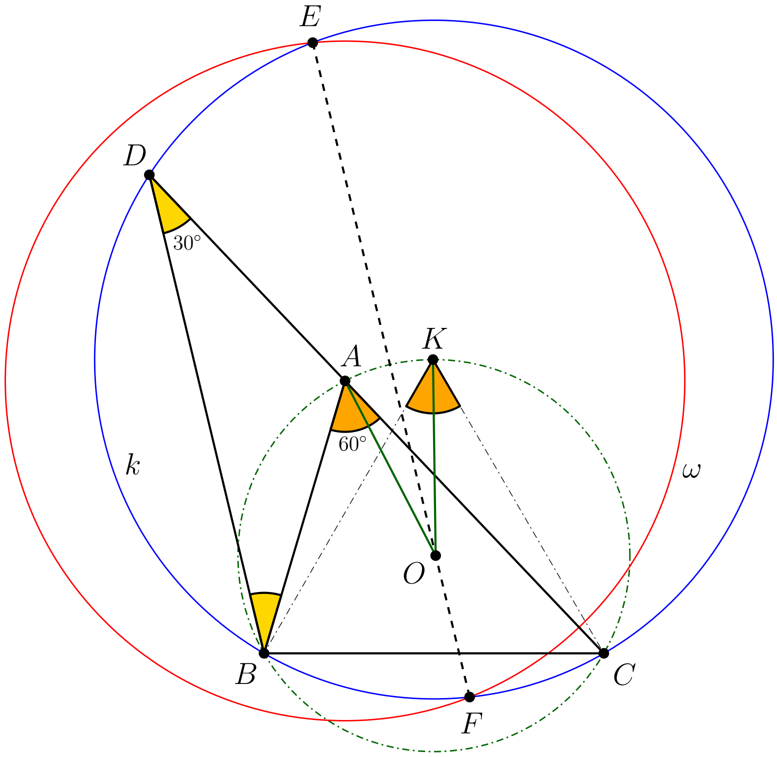
30°. This has two important corollaries. Firstly, by inscribed angles $\angle BKC = 60^\circ = \angle BAC$, so it follows that K lies on the circumcircle of triangle ABC (as K and A lie on the same side of line BC). In fact, $OA = OK$. Secondly, the radius of k is

$$\frac{1}{2} \cdot \frac{BC}{\sin \angle BDC} = \frac{BC}{2 \sin 30^\circ} = BC.$$

Therefore,

$$\mathrm{Pow}_k(O) = OK^2 - (\text{radius of } k)^2 = OK^2 - BC^2 = OA^2 - BC^2 = OA^2 - (\text{radius of } \omega)^2 = \mathrm{Pow}_\omega(O).$$

This finishes the proof.



T-6

Let ABC be an acute triangle. Let M be the midpoint of the segment BC. Let I, J, K be the incenters of triangles ABC, ABM, ACM, respectively. Let P, Q be points on the lines MK, MJ, respectively, such that $\angle AJP = \angle ABC$ and $\angle AKQ = \angle BCA$. Let R be the intersection of the lines CP and BQ. Prove that the lines IR and BC are perpendicular.

Solution. Note that $MK \perp MJ$. By simple angle chasing we get that

$$\angle PJM = \angle AJM - \angle AJP = 90^\circ + \frac{1}{2}\angle ABC - \angle ABC = 90^\circ - \frac{1}{2}\angle ABC,$$

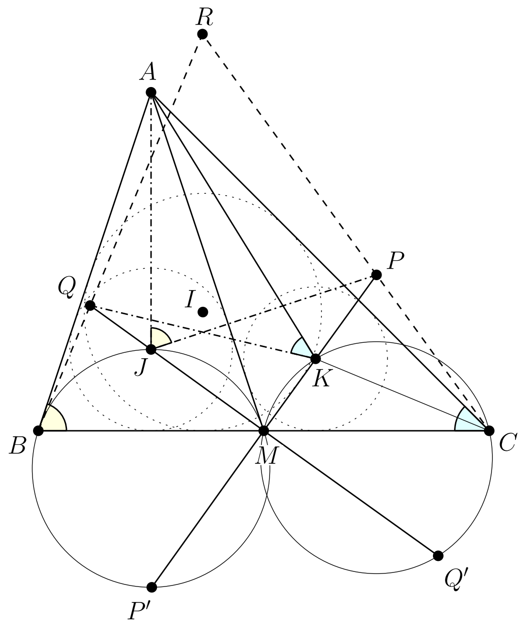
$$\angle JPM = 90^\circ - \angle PJM = \frac{1}{2}\angle ABC.$$

Let P' be the reflection of P over M. Note that $\angle JPM = \angle JP'M$ as $JM \perp MP$. Then

$$\angle JP'M = \angle JPM = \frac{1}{2}\angle ABC = \angle JBM,$$

hence $JBP'M$ is concyclic and as $MK \perp MJ$, point P' is the A-excenter of triangle ABM by the incenter-excenter lemma. Let Q' be the reflection of Q over M. Analogously, we get that Q' is the A-excenter of triangle AMC.

Let R' be the intersection of lines BP' and CQ'. Note that R' is the A-exceneter of triangle ABC. Point R is the reflection of R' over M as by reflecting lines BP', CQ' over M we get lines CP, BQ. It is well-known that distance of incenter and the distance of A-excenter from the perpendicular bisector of BC is the same, thus by reflecting R' over M we get a point (R) that lies on the line through I perpendicular to BC.

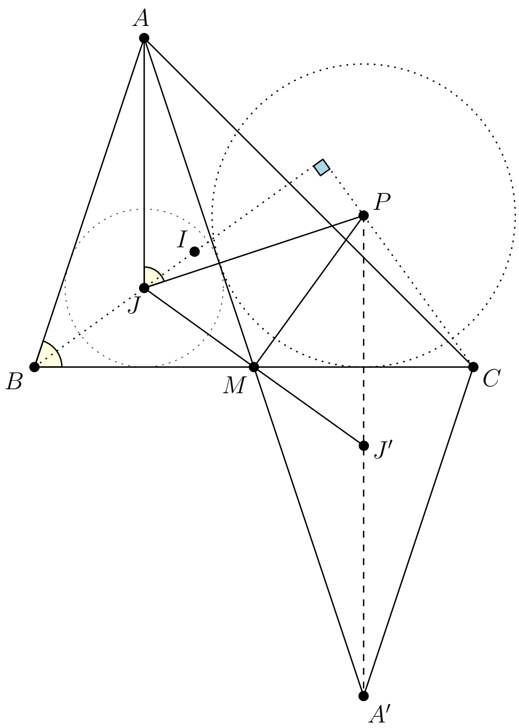


Comment. Another way to finish the problem is by using $BP' \parallel CP$ and $CQ' \parallel BQ$ to show that I is the orthocenter of triangle RBC, as $IB \perp BP'$ and $IC \perp CQ'$.

Solution 2. Let A' and J' be the reflections of A and J in M, and let us denote $\angle ABC$ by β.

First we show that A', J' and P are collinear. Obviously, $\angle JMP = 90°$, which implies that $\angle PJJ' = \angle PJ'J$. Simple angle chasing shows that $\angle MJA = 90° + \beta/2$. Since $\angle AJP = \beta$, we obtain $\angle MJP = 90° - \beta/2$, and by the reflection it follows that $\angle MJ'A' = 90° + \beta/2$ and $\angle MJ'P = 90° - \beta/2$. This yields that A', J', and P are collinear.

Now we observe that P is the intersection of two angle bisectors, thus P is the A'-excenter of $MA'C$. It follows that BI and CP are two perpendicular angle bisectors. One can show similarly that $CI \perp BQ$, hence I is the orthocenter of BCR, and the statement follows.



Solution 3. It is enough to show that I is the orthocenter of triangle RBC. By symmetry, it suffices to prove that $BI \perp RC$. Denote the midpoints of sides AB, AC by B' and C', respectively. The famous Iran lemma tells us that the projection of C onto line BI lies on MC'. Thus, we need to prove that lines BI, MC', CP are concurrent. This is the same as saying that triangles BCC', JPM are perspective, which - by Desargues's theorem - is equivalent to the points $BC \cap JP, CC' \cap PM, C'B \cap MJ$ being collinear.

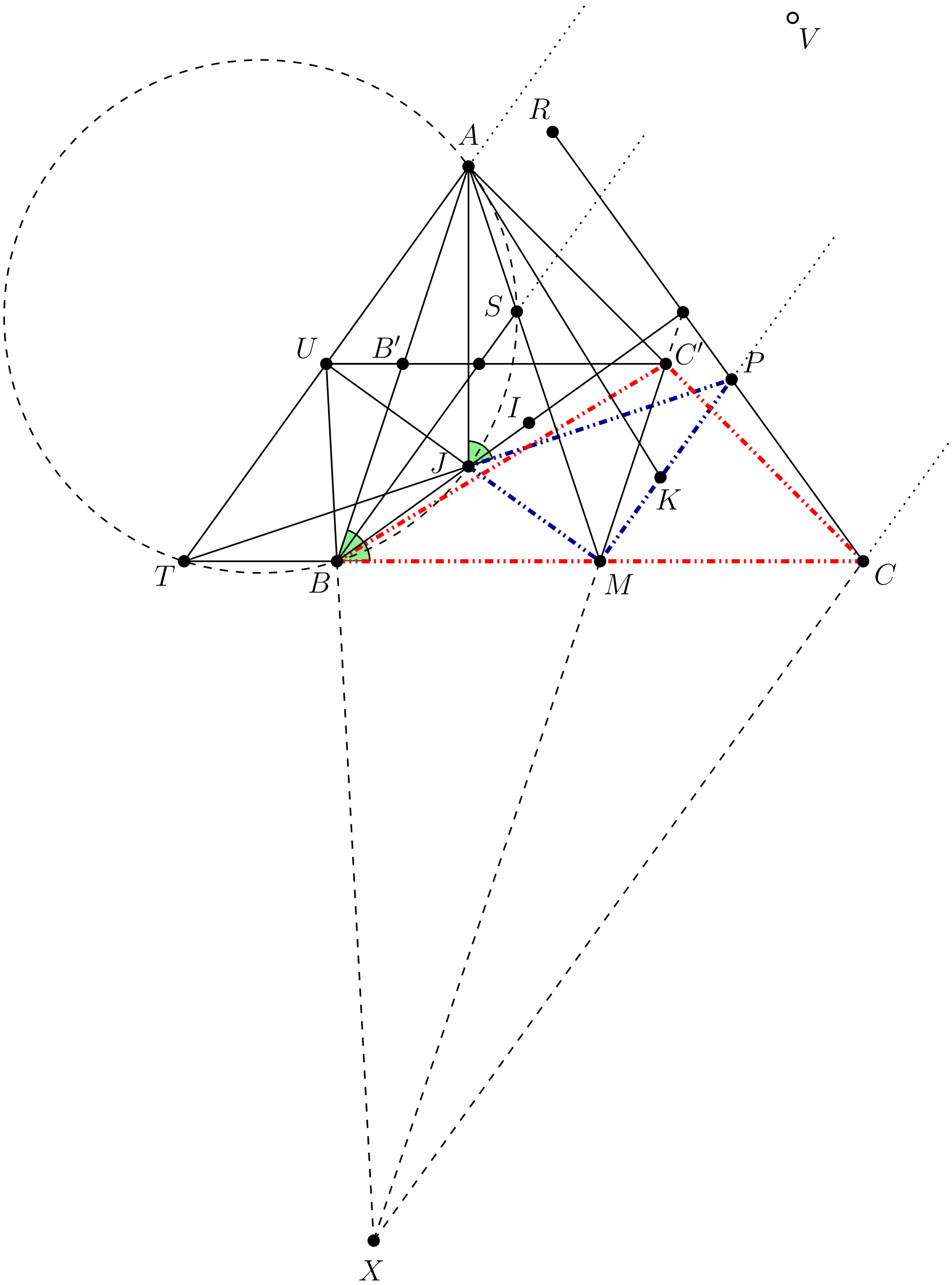
Now let us define some new points. Let the circumcircle of AJB intersect lines AM, BM for the second time at S and T, respectively. Since this circle is symmetric with respect to MJ, and so are the lines AM, BM, we have that $ASBT$ is an isosceles trapezoid. Notice that the angle condition of the problem tells us

$$\angle AJT + \angle AJP = \angle ABT + \angle AJP = \angle ABT + \angle ABC = 180^\circ,$$

thus T, J, P are collinear. Now observe that $\angle JMK = \angle JMA + \angle KMA = \angle BMA/2 + \angle CMA/2 = 180^\circ/2 = 90^\circ$. Also, as MJ is the perpendicular bisector of AT and BS, we have that lines MK, AT, BS are all perpendicular to MJ, so they are all parallel.

Let U be the midpoint of AT and V be the point at infinity of line MK. As we saw previously, U lies on MJ and V lies on AT. Now, $BC \cap JP = BC \cap UV, CC' \cap PM = CC' \cap VM, C'B \cap MJ = C'B \cap MU$. We wish to show that these points are collinear, which now becomes same as saying that triangles BCC', UVM are perspective. By Desargues's theorem, it suffices to prove that lines $BU, CV, C'M$ are concurrent.

Suppose that lines $BU, C'M$ intersect at X. Notice that U, B', C' all lie on a midline of triangle ABC and $B'B \parallel C'M$. It follows that triangles $BMX, UB'B$ are homothetic, so it is enough to prove our goal for triangle $UB'B$. That is, we need to show that if U' is the reflection of U in B', then BU' is parallel to BS. However, this is fairly trivial, as B' lies on the midline of the isosceles trapezoid $ASBT$ (since it is the midpoint of diagonal AB), so U' must lie on BS. We are finally done.



Comment. We used the fact that $S \neq A$ and $T \neq B$. If this wasn't the case, then we would have $AM = BM = CM$, which would lead to a $90°$ angle at C in triangle ABC, contradicting the acuteness of the triangle.

T-7

Define *glueing* of positive integers as writing their base ten representations one after another and interpreting the result as the base ten representation of a single positive integer.

Find all positive integers k for which there exists an integer N_k with the following property: for all $n \geq N_k$, we can glue the numbers $1, 2, \ldots, n$ in some order so that the result is a number divisible by k.

Remark. The base ten representation of a positive integer never starts with zero.

Example. Glueing $15, 14, 7$ in this order makes 15147.

Answer. k has this property if and only if $3 \nmid k$.

Solution. If $3 \mid k$, then for any $n \in \mathbb{N}$ of the form $n = 3k + 1$, the sum of the digits of all positive integers up to n gives remainder 1 modulo 3. As a positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3, no matter how we glue $1, 2, \ldots, n$, the resulting number is not divisible by 3, hence also not by k. So no appropriate N_k can be chosen.

If $3 \nmid k$, write $k = 2^a 5^b \cdot m$ with $\gcd(m, 10) = 1$. Let $\ell \geq \lfloor \log_{10}(m) \rfloor + 2$ be an integer divisible by $\phi(m)$, so $10^\ell \equiv 1 \pmod{m}$ by Euler-Fermat. By the size constraint of ℓ, we can choose $a_i \in \mathbb{N}$ for $0 \leq i < m$ such that $a_i \equiv i \pmod{m}$ and a_i has ℓ digits. Note that all a_i are different and can also be chosen to be different from $10^{\max(a,b)}$. Take $N_k \geq \max(10^{\max(a,b)}, 10^l)$. For $n \geq N_k$ we glue $1, 2, \ldots, n$ the following way:

(a) We put $10^{\max(a,b)}$ at the end of the glueing, so the glued number ends with $\max(a, b)$ zeros, ensuring that the number is divisible by $2^a 5^b$.

(b) At the beginning of the number we put a_0, then the number 1, then $a_1, a_2, \ldots, a_{m-1}$ in this order.

(c) We put the remaining numbers in the middle, in an arbitrary order.

Let G_0 be the number given by this glueing, and for $1 \leq i < m$ let G_i by the number we get from the same glueing, except for swapping a_0 and a_i. We show that at least one of the integers $G_0, G_1, \ldots, G_{m-1}$ is divisible by m, and as all of them are divisible by $2^a 5^b$, one of them will be divisible by k, finishing the proof.

Let d be the number of digits of G_0 (and hence of $G_1, \ldots, G_{m-1}$ as well). Then

$$G_i - G_0 = 10^{d-\ell}(a_i - a_0) + 10^{d-\ell-1-\ell \cdot i}(a_0 - a_i) \equiv 10^{d-1} \cdot (a_i - a_0) \cdot (10 - 1) \equiv i \cdot 9 \cdot 10^{d-1} \pmod{m}.$$

So if i is such that $-G_0 \equiv i \cdot 9 \cdot 10^{d-1} \pmod{m}$, then $m \mid G_i$. This choice of i is possible, as m is coprime to both 9 and 10, so the inverse of $9 \cdot 10^{d-1}$ exists modulo m. Therefore G_i is a gluing of $1, 2, \ldots, n$, divisible by k.

T-8

Let k be a positive integer and $a_1, a_2, \ldots$ be an infinite sequence of positive integers such that

$$a_i a_{i+1} \mid k - a_i^2$$

for all integers $i \geq 1$. Prove that there exists a positive integer M such that $a_n = a_{n+1}$ for all integers $n \geq M$.

Solution 1. Note that $a_i \mid k$ for all $i \geq 1$. Furthermore, we have $a_{i+1} \mid a_i^2$, so there are only finitely many primes that divide any element of the sequence.

For a prime number p and a positive integer n, let $\nu_p(n)$ denote the exponent of p in the prime factorization of n.

We'll prove that $\nu_p(a_{i+1}) \leq \nu_p(a_i)$ for all primes p and all but finitely many i. From this, the claim will follow as for each prime p, the sequence $(\nu_p(a_i))_i$ is eventually constant, and there are finitely many primes p to consider.

Take a prime number p. Suppose that $\nu_p(a_{i+1}) > \nu_p(a_i)$ for some positive integer i (if i with this property don't exist, we're done). Since $a_i a_{i+1} \mid k - a_i^2$, we must have $\nu_p(a_i^2) = \nu_p(x)$, as otherwise $\nu_p(a_i^2 - x) \leq \nu_p(a_i^2) < \nu_p(a_i a_{i+1})$, a contradiction.

Then $a_{i+1}a_{i+2} \mid k - a_{i+1}^2$, and since $\nu_p(x) = \nu_p(a_i^2) < \nu_p(a_{i+1}^2)$, we have $\nu_p(x - a_{i+1}^2) = \nu_p(x)$, and $\nu_p(a_{i+1}a_{i+2}) \leq \nu_p(x)$, from where it follows that $\nu_p(a_{i+2}) < \frac{\nu_p(x)}{2}$.

Now $a_{i+2}a_{i+3} \mid k - a_{i+2}^2$ and from $\nu_p(a_{i+2}) < \frac{\nu_p(x)}{2}$ we have that $\nu_p(x - a_{i+2}^2) = \nu_p(a_{i+2}^2)$, therefore $\nu_p(a_{i+3}) \leq \nu_p(a_{i+2}) < \frac{\nu_p(x)}{2}$.

Repeating the same argument for $i+3, i+4, \ldots$ gives us

$$\nu_p(a_{i+j}) \leq \nu_p(a_{i+j-1}) \leq \frac{\nu_p(x)}{2}$$

for $j \geq 3$, and we're done.

Solution 2. We can finish the proof slightly differently. We have already seen that there are only finitely many primes dividing any element of the sequence. So it is enough to prove that for any such prime p there is M such that $\nu_p(a_n)$ is constant for any $n \geq M$. Take any such prime p.

Suppose that there is i such that $\nu_p(a_j)$ takes its minimum, that is $\nu_p(a_i) \leq \nu_p(a_j)$ for all j, furthermore $\nu_p(a_{i+1}) > \nu_p(a_i)$. (If there is no such i, then we have proved the required property for p.)

We know that $a_i a_{i+1} \mid k - a_i^2$ and $a_{i+1} a_{i+2} \mid k - a_{i+1}^2$. Then from the fact that $\nu_p(a_i)$ is minimal we know that $\nu_p(x - a_{i+1}^2) \geq \nu_p(a_{i+1}) + \nu_p(a_i)$. Therefore

$$\nu_p(a_{i+1}) + \nu_p(a_i) \leq \nu_p((x - a_{i+1}^2) - (x - a_i^2)) = \nu_p(a_i + a_{i+1}) + \nu_p(a_{i+1} - a_i) = 2\nu_p(a_i).$$

But then we get that $\nu_p(a_{i+1}) \leq \nu_p(a_i)$, which is a contradiction.